Abstract: We consider a dynamic pricing problem for a monopolistic company selling a perishable product when customer demand is uncertain and occurs in batches which must be fulfilled as a whole. The seller can price-discriminate between the batches of different size by setting different unit prices. The problem is modeled as a stochastic optimal control problem to find an inventory-contingent dynamic pricing policy to maximize the expected total revenues. We find the optimal pricing policy and prove several monotonicity results. First, we establish stochastic order conditions on the unit willingness-to-pay distributions that determine when quantity discounts or premiums take place as a batch purchase is broken up into a rapid sequence of smaller purchases. Second, we give a sufficient condition for prices to be monotonically decreasing in inventory when the maximum batch size is two. Third, we establish sufficient stochastic order and value function convexity conditions for the perceived quantity discounts and premiums which result from comparing unit prices for different batch sizes under a particular inventory level. Numerical experiments reveal further details of interactions between stochastic order and convexity conditions, and demonstrate that a company can generate significant additional revenues through explicit dynamic non-linear pricing.

Keywords: dynamic pricing, batched demand, stochastic orders, dynamic programming applications

1. Introduction

Dynamic pricing and other revenue management (RM) practices are becoming crucial in the operation of many industries including travel, hospitality, entertainment, and other services. To a great extent, this process is facilitated by the development of e-commerce that permits the sellers to dynamically adjust prices based on the current market environment and various inputs from potential buyers including a purchase size. In static pricing, charging different unit prices depending on the purchase size is referred to as nonlinear pricing. If the unit prices also vary with time, we talk about dynamic nonlinear pricing (DNP).

Customers are generally familiar with the practice of nonlinear pricing in retail. Examples include charging different unit prices for packages of different sizes and promotional strategies such as “buy two, get one free” or “buy one, get the second 50% off”. Bundling may also be viewed as a form of nonlinear pricing that involves multiple product types. However, in brick-and-mortar retail stores, pricing cannot be fully dynamic because the prices are typically fixed for a period of time. Customers can obtain inventory information by going to the store and, in their encounters with nonlinear prices, expect quantity discounts: a unit price that decreases with the purchase size. Indeed, an increasing unit price would simply result in customer making separate smaller purchases. In principle, it is possible to implement quantity premiums but this would require limiting the number of purchases per customer within the time interval in which prices are fixed and, as a result, may alienate customers. Thus, quantity premiums in retail are almost never observed, except in situations that rely on the lack of customer attention to unit prices. Online retailers that rely on posted prices are also subject to customer expectations of
<table>
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<td>5</td>
<td>$787.19</td>
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Table 1. Price per ticket on WestJet roundtrip YYZ-CUN departing on Sep. 17, 2011 and returning on Sep. 24, 2011 and corresponding quantity premium as % of a single ticket price

quantity discounts. However, are such expectations generally justified in e-commerce systems where prices may vary among individual transactions?

Experience with online travel booking sites reveals that quantity premiums do occur. A recent search on Expedia.ca for a WestJet round-trip fare on a direct flight from Toronto to Cancun has revealed a startling 32% increase in unit prices as the number of seats required by the search increased from one to five (see Figure 1). The figure shows that there were several increases in unit price, indicating that WestJet employed a nonlinear pricing policy with quantity premiums. Do such premiums go against our intuition that customers usually expect discounts?

We argue that in the context of an electronic dynamic pricing system the answer is “not necessarily”. First, traditional yield management systems with nested booking limits automatically enforce quantity premiums since each subsequent seat can only become more expensive for the customer as consecutive booking limits are getting filled. Second, it may actually be optimal for the firm to introduce premiums when they are required by the demand structure. In this paper, we seek to describe the demand conditions under which the firm should employ quantity premiums or discounts. Markets where this is especially important are characterized by customer segmentation according to the purchase size. While each customer may potentially require a different amount of the product, there are many contexts in which this amount is fixed before the customer makes a purchase decision. Examples include airline or hotel bookings where the size of the group is fixed, group purchases of event tickets, and group admissions to recreational facilities (e.g., golf courses). Barriers (also called “fences”) for customer migration among the size segments work only in one direction; customers can always substitute a larger batch if its price is lower, but cannot substitute a smaller one. This forces an order constraint on feasible prices.

The pricing mechanisms in the above WestJet example belong to a broadly defined category of dynamic pricing for perishable products with uncertain demand segmented according to the batch size. Quantity premiums and discounts in this context are a type of monotonic property of the pricing policy. In this paper, we study three classes of such properties. Monotonicity of prices as functions of the remaining inventory is usually considered in the literature. For example, Gallego and van Ryzin (1994) described the conditions under which the optimal price for one unit of a perishable product decreases in the remaining inventory. A second type of monotonicity can only occur when we consider demand for batches of different size. We refer to this type as a break-up monotonicity, that is, a monotonic relation of the batch price relative to the sum of prices of sub-batches purchased in rapid succession. If the batch price is higher than the sum of prices of sub-batches, customers may have an incentive to circumvent the pricing policy of the firm. Although, if customers do not have full information about the pricing policy and there is sizable
supply risk, this incentive is significantly reduced. The third type of monotonicity is a relation between unit prices for batches of different size at the same inventory level. It can be described as \textit{perceived} quantity discounts or premiums. The example presented in Figure 1 represents a monotonic property of this type (the perceived quantity premium), since we do not know how the price would actually behave had we purchase the tickets.

In the analysis of the proposed model, we obtain the following insights. First, under a mild restriction that customer willingness-to-pay (WTP) per batch is increasing (stochastically in the likelihood ratio order) with respect to the batch size, we establish that the optimal dynamic pricing policy is characterized by the property that the marginal revenue rate for each batch size is equal to the opportunity cost associated with selling this batch. Ordering of the WTP distributions permits to effectively eliminate the order constraint on batch prices.

Second, we use this characterization to analyze each of the three types of monotonic properties. Break-up monotonicity of prices is determined by the interaction of the order in WTP per unit of the batch and convexity property of the reciprocal hazard rate of the WTP distribution. In particular, we prove that when customer WTP \textit{per unit} of the batch is increasing (stochastically in the hazard rate order) with respect to batch size and the reciprocal of the hazard rate is concave, then there is a quantity premium for the batch purchased as a whole compared to smaller batches purchased in rapid succession and totaling the same number of units. On the other hand, when WTP per unit of the batch is decreasing with respect to batch size and the reciprocal of the hazard rate is convex, then there is a quantity discount (again relative to a break-up of the batch). For the exponential WTP distribution (which has a constant hazard rate) this result provides a complete characterization of the break-up monotonicity property.

Third, monotonicity of prices with respect to the remaining inventory is linked to the convexity properties of the value function of the problem (i.e., the optimal expected revenues given a particular remaining time and inventory level). However, neither concavity nor convexity of the value function can be guaranteed in general. It is easy to construct a problem instance where neither property will hold. For the special case of the maximum batch size of two, we present a sufficient condition that guarantees the concavity of the value function. Convexity and concavity of the value function also play a role in relations between prices for batches of different size (perceived quantity discounts or premiums). In particular, if the value function is concave and the customer WTP \textit{per unit} of the batch is increasing (stochastically in the hazard rate order) with respect to batch size then there are perceived quantity premiums. On the other hand if the value function is convex and the customer WTP \textit{per unit} is decreasing then there are perceived quantity discounts. We resort to numerical experimentation to further refine these insights.

In particular, numerical experiments suggest that when customer WTP per unit of a batch is much smaller than for a single unit then, regardless of the concavity properties of the value function, the policy will exhibit perceived quantity discounts. Thus, the direct price-reducing effect of the low batch unit WTP is much stronger than its opposing indirect effect which leads to concavity of the value function. On the other hand, when the batch unit WTP is high, the presence of quantity premiums may be precluded by the convexity of the value function. That is, an indirect effect may dominate.

Another important issue is the value of the explicit DNP policy compared to plausible “legacy” approaches which do not take into account customer segmentation according to batch size. When there is a pattern of significant demand for large batches, the benefits of the explicit DNP can be very significant, especially if customer WTP per unit is stochastically increasing in the batch size. Rather conservative settings examined in the numerical experiments show the benefits of
up to 13% of the expected revenues. However, even when customer WTP per unit is decreasing, the increase in revenue may exceed 1%.

As a final note, we hope that the proposed analytic techniques and structural results (in particular, break-up monotonicity) can be generalized to provide insights for the dynamic pricing problem in network RM.

The organization of the paper is as follows. In §2 we position our work within the existing literature. The model is presented in §3, while the characterization of the optimal dynamic pricing policy and its monotonic properties are studied in §4. We present numerical experiments in §5. We also discuss a deterministic approximation to the DNP problem in Appendix §B.

2. Literature

Nonlinear pricing is a widespread practice with areas of application including such important fields as international trade, telecommunications, transportation, energy, supply chains, and retail. Wilson (1993) provides an overview of the massive economics and marketing literature on this subject. A classical approach to the problem in this literature is via the theory of incentives or mechanism design (see, for example, Laffont and Martimort (2001)). In these fields, the focus is usually not on the limited capacity or dynamic aspects of the problem that are critical to revenue management. Dhebar and Oren (1986) consider a dynamic model of pricing for a new product whose consumption value increases as the market expands (for example, telecommunications). Starting with the classical work of Gallego and van Ryzin (1994), dynamic pricing of limited capacity over a finite horizon is one of the main approaches to RM; see Chapter 5 of Talluri and van Ryzin (2004). Gallego and van Ryzin (1994) and subsequent works analyze the optimality conditions and the monotonic properties of the optimal pricing policy in various variants of this general setting. A typical assumption in this stream of literature is that customers demand one unit of the product at a time. One exception is Elmaghraby et al. (2008) who consider the design of optimal markdown mechanisms when rational customers have multi-unit demands. However, in that model the number of customers and their demands are known in advance and there is no possibility to explicitly price-differentiate between customers who require different sizes. Dynamic nonlinear pricing can also be viewed as dynamic pricing for multiple products using a shared resource capacity. Such models were considered in the RM literature, for example, by Maglaras and Meissner (2006), who analyze a general model where each product uses one unit of a shared resource (unlike our case, in which the size variation between products is critical). The article presents a unified treatment of dynamic pricing and capacity control formulations, a reduction of the problem to the one based on the aggregate consumption rate, and asymptotically optimal heuristics based on the deterministic approximation. Some articles (for example, Zhang and Cooper (2009), Dong et al. (2009)) consider dynamic pricing of substitutable perishable products. A distinguishing feature of these works is that customers can choose between different products with independent resources. This is different from a setting (as in our paper) where the choice of the product is fixed in advance and the capacity is shared.

On the other hand, nonlinear pricing is naturally embedded into the network RM models. In the general model of Gallego and van Ryzin (1997), demand requests of different size are represented as products with different capacity requirements. However, since the focus of that article is on the general network model, it does not discuss the monotonic structure of the optimal pricing policy which is required to answer the questions posed in our work. Gallego and van Ryzin (1997) also do not consider batch price order constraints that arise because a larger batch is always substitutable for a smaller one.
To develop our analysis, we build on the technical approach employed by Gallego and van Ryzin (1994, 1997) which views dynamic pricing as intensity control of a stochastic point process. To this, we add the analysis of problem inputs by means of stochastic order relations between customer willingness-to-pay random variables. For a review of stochastic orders, see Shaked and Shanthikumar (1994).

3. Model

We consider a problem of optimal dynamic pricing of a fixed stock of \( y \) identical perishable items over a time horizon of length \( t \) (using a reversed time index, i.e., \( t \) is the beginning and \( 0 \) is the end of the horizon). Customer demand is uncertain and occurs in batches that must be fulfilled as a whole. Thus, the seller can price-discriminate between the batches of different size by using different unit prices. Customer quote requests for batches of \( i = 1, \ldots, y \) items arrive according to independent counting processes \( N_{it} \) with constant intensities \( \lambda_i \). Upon request, the firm immediately quotes batch price \( p_i \), and the purchase is completed by the customer with probability \( \pi_i(p_i) \). Resulting batch purchases of size \( i \) are described by “thinned” counting process \( \tilde{N}_{it} \) whose intensity \( \lambda_i \pi_i(p_i) \) is price-modulated.

3.1. Discussion of problem inputs: \( \lambda_i \) and \( \pi_i(p_i) \). Functions \( \pi_i(p_i) \), \( i = 1, \ldots, y \) can be viewed as complementary cumulative distribution (survival) functions of given random variables as well as appropriately normalized aggregate demand functions. The latter are commonly assumed to be regular in the dynamic pricing literature (see, Assumption 7.1 in Talluri and van Ryzin (2004)). Conditions (i)-(iv) of Assumption 7.1, in a probabilistic interpretation, can be expressed as follows:

**Assumption 1.** Each function \( \pi_i(p_i) \), \( i = 1, \ldots, y \) is the complementary cumulative distribution (survival) function of the willingness-to-pay (WTP) random variable \( W_i^b \) for a size \( i \) batch. \( W_i^b \) is an absolutely continuous nonnegative random variable whose support set is an entire half-line.

The condition on support sets makes this assumption somewhat stronger than Assumption 7.1 but it is not very restrictive. An immediate consequence is that the survival function has the inverse \( p_i = \pi_i^{-1}(d_i) \) which is a strictly decreasing continuously differentiable function of purchase probability (demand) \( d_i \) (for all \( d_i > 0 \)). Moreover, it is possible to express the expected batch revenues \( r_i(p_i) = p_i \pi_i(p_i) \) per request as a function of \( d_i \). We use this fact to express the problem in terms of \( d_i \)’s as decision variables. With a slight abuse of notation, we use \( r_i(d_i) = \pi_i^{-1}(d_i)d_i \) to denote the expected batch revenues as a function of batch purchase probability \( d_i \).

It is important to understand how (WTP) random variables \( W_i^b \) and the arrival rates \( \lambda_i \) arise in the process of segmentation of customer population according to batch size. Let \( \Theta \) be a set of all possible customer types \( \theta \) (in some appropriate parameter space) and \( \phi(\theta) \) be a distribution of customers in the population over types. Suppose that each customer type \( \theta \in \Theta \) has a specific WTP value \( v_i(\theta) \) for the batch of size \( i \), and that type \( \theta \) demands size \( i \) with probability \( \psi_i(\theta) \). The collection of functions \( \psi_i(\theta) \), \( i = 1, \ldots, y \), in effect, describes the segmentation mechanism of customer population by the batch size and, together with \( \psi_0(\theta) = \sum_{i=1}^y \psi_i(\theta) \), defines a probability mass function over \( \{0, 1, \ldots, y\} \) parameterized by \( \theta \). This setup can be illustrated by a group of friends who periodically travel to a vacation spot such as Cancun. Friends have a preference for traveling together and may place a group booking as a result. Valuations for different batch sizes are consistent over different trips but the size may vary if not all members...
of the group can travel each time. The fraction of all population members who demand size $i$ is
\[ \Psi_i = \int_{\Theta} \psi_i(\theta) \phi(\theta) d\theta. \] (1)

If we let $\lambda$ be an exogenous arrival rate for all batch sizes combined, then the arrival rate for each size $i$ is $\lambda_i = \lambda \Psi_i$. Under very mild technical assumptions on $v_i(\theta)$, the survival function of $W^b_i$ is given by
\[ \pi_i(p) = \frac{1}{\Psi_i} \int_{v_i(\theta) \geq p} \psi_i(\theta) \phi(\theta) d\theta. \] (2)

The representation of the inputs given by (1) and (2) is very general. For example, consider customer types associated with WTP vectors. Then, any given $\lambda_i$ and $\pi_i(p), i = 1, \ldots, y$ can be represented by the set of types $\Theta = \mathbb{R}^y_+$, overall arrival rate $\lambda = \sum_{i=1}^y \lambda_i$, constant size realization probability $\psi_i(\theta) = \frac{\lambda_i}{\lambda}$ (independent of type $\theta$) and distribution over customer types $\phi(\theta) = \prod_{i=1}^y (-\pi_i'(\theta_i))$.

3.2. Dynamic nonlinear pricing as an intensity control problem. In presenting the formal description of the problem, we start with the set of the feasible pricing policies. Feasible batch prices have to be constrained so that they are increasing in the batch size and customers truthfully reveal the required batch size. Indeed, if there is ever a violation to this increasing order, customers can “jump the fence” between the size segments risk-free and purchase a batch of the larger size than they actually need. Thus, all feasible price vectors must belong to the set
\[ P^y = \{ p \in \mathbb{R}^y_+: p_{i+1} \geq p_i, \ i = 1, \ldots, y - 1 \}. \] (3)

Equivalently, feasible purchase probabilities must belong to the set
\[ D^y = \{ d \in [0, 1]^y: \pi_{i+1}^{-1}(d_{i+1}) \geq \pi_i^{-1}(d_i), \ i = 1, \ldots, y - 1 \}. \] (4)

For batches exceeding the available capacity, we use a standard convention that their prices are forced to $\infty$, effectively shutting down the demand since $\pi_i(p_i) \to 0$ as $p_i \to \infty$. Both this and an equivalent condition $d_i = 0$ for $i$ in excess of the current capacity can be expressed by the constraint
\[ \sum_{i=1}^y \int_0^t d\tilde{N}_{it} \leq y \ (\text{a.s.}). \] (5)

Now, let $U^y$ be the set of all nonanticipating policies $p_t \in P^y$ (equivalently, $d_t \in D^y$) which satisfy (5). The problem of the firm is to find a control policy $u \in U^y$ such that the expected total revenue
\[ V^u(t, y) = \mathbb{E}_u \left[ \sum_{i=1}^y \int_0^t p_{iu} d\tilde{N}_{it} \right] \] (6)
is maximized. The optimal value, if it exists, is denoted as $V(t, y) = \sup_{u \in U} V^u(t, y)$.

The stated problem is that of intensity control, with intensities expressed via purchase probabilities $d_i$ as $\lambda_i d_i$. Sufficient optimality conditions to this problem are given by the Hamilton-Jacobi-Bellman equation (see Theorem VII.1 of Bremaud (1981)):
\[ \frac{\partial}{\partial t} V(t, y) = \sup_{d \in D^y} \left\{ \sum_{i=1}^y \lambda_i [r_i(d_i) - d_i(V(t, y) - V(t, y - i))] \right\}, \ \forall \ y \geq 1, \ t \geq 0, \] (7)
with the boundary conditions
\[ V(0, y) = 0, \quad \forall y \geq 1, \quad (8) \]
\[ V(t, 0) = 0, \quad \forall t \geq 0. \quad (9) \]

In equation (7), the dimension of the feasible decision space \( \mathcal{D}^y \) automatically ensures that demand for batch sizes higher than the available capacity is shut down. The following condition (which implies part (v) of Assumption 7.1) is sufficient for the supremum to be attained in equation (7) by a non-trivial maximizer:

**Assumption 2.** \( \lim_{d_i \to 0} r_i(d_i) = 0, \quad i = 1, \ldots, y. \)

This condition implies that each \( r_i(d_i) \) is continuous at 0 and, therefore, continuous on the entire interval \([0, 1]\) (continuity for \( d_i > 0 \) follows from continuous differentiability of \( \pi_i^{-1}(d_i) \)). As a consequence, batch revenues \( r_i(d_i) \) have maximizers \( 0 < d_i^* < 1, \quad i = 1, \ldots, y \) and the corresponding price \( p_i^* = \pi_i^{-1}(d_i^*) > 0 \) is finite. Continuity of each \( r_i(d_i) \) implies continuity of the expression under the supremum in (7). This, along with the observation that \( \mathcal{D}^y \) is compact, implies that the inputs to the problem satisfy conditions of Theorem VII.3 of Bremaud (1981) which claims the existence of the unique solution to (7) and a corresponding Markovian optimal policy \( d_i(t, y) \). However, the form of the optimal policy is fairly generic at this point and its properties are unclear. In the next section, we investigate additional assumptions on problem inputs, and study resulting properties of the optimal policy.

### 4. Properties of the optimal policy

#### 4.1. Characterization of the optimal pricing policy

We begin the analysis by deriving a compact characterization of the optimal policy. This characterization serves as a foundation for the analysis of the monotonic properties of the optimal policy which we discuss in the following subsections. The main technical condition used for the proposed characterization can be interpreted as an intuitive statement that “customers in segments corresponding to larger batch sizes generally have a higher willingness to pay”.

To guarantee that the first-order optimality condition is sufficient for the maximum in single-product pricing problems, it is frequently assumed that the revenue functions \( r_i(d_i) \) are strictly concave; see, for example, Assumption 7.2 of Talluri and van Ryzin (2004). We make a somewhat stronger but not very restrictive assumption:

**Assumption 3.** For each \( i = 1, \ldots, y \), hazard rate \( h_i(p_i) = \frac{f_i(p_i)}{\pi_i(p_i)} \) of random variable \( W_i^b \) is increasing.

This assumption guarantees that \( r_i(d_i) \)'s are strictly concave. To see this, consider the marginal revenue rates
\[ J_i(d_i) = \pi_i^{-1}(d_i) + d_i(\pi_i^{-1})'(d_i), \quad (10) \]
and their re-parameterizations in terms of prices
\[ J_i(p_i) = p_i - (h_i(p_i))^{-1}, \quad (11) \]
(similar definitions are used in Talluri and van Ryzin (2004)). The revenue function \( r_i(d_i) \) is strictly concave if and only if \( J_i(d_i) \) is strictly decreasing, and, in turn, if and only if \( J_i(p_i) \) is strictly increasing. Increasing (not necessarily strictly) \( h_i(p_i) \) is sufficient for the latter to hold. Assumption 3 is not very restrictive because increasing hazard rate is equivalent to log-concavity of the survival function which is true for any log-concave density (exponential, normal, gamma,
etc). However, the univariate concavity assumption is no longer sufficient in the multivariate setting since the feasible set $D^y$ is not necessarily convex. Therefore, to ensure the convexity of $D^y$ we make the following additional

**Assumption 4.** The ratio $\frac{f_i(p)}{f_{i+1}(p)}$ is decreasing.

This condition defines a likelihood ratio order between $W^b_i$ and $W^b_{i+1}$, that is $W^b_i \leq_{lr} W^b_{i+1}$ according to the definition in §1.C.1 of Shaked and Shanthikumar (1994). This relation makes intuitive sense since customers are usually prepared to pay more for larger batches. The convexity claim is summarized as the following

**Lemma 1.** Under Assumptions 1 and 4, set $D^y$ of feasible purchase probabilities is convex.

Likelihood ratio order $W^b_i \leq_{lr} W^b_{i+1}$ implies hazard rate order $W^b_i \leq_{hr} W^b_{i+1}$; see Shaked and Shanthikumar (1994). That is, $h_i(p) \geq h_{i+1}(p)$ for all $p$. This condition leads to the following useful observation, summarized as a lemma (proof is immediate):

**Lemma 2.** Under Assumption 1 along with condition $W^b_i \leq_{hr} W^b_{i+1}$, we have $J_i(p) \geq J_{i+1}(p)$.

The characterization of the unique optimal policy is summarized as the following proposition which shows that the solution to the univariate optimality conditions with respect to each $d_i$ is not only feasible but also optimal for the multivariate problem. Thus, our assumptions on the input data automatically enforce monotonicity relations embodied in the definition of the feasible set $P^y$.

**Proposition 1.** Under Assumptions 1-4, the pointwise optimization subproblem in (7) has a strictly concave objective function and a convex feasible set, and there exists the unique optimal policy determined by the univariate first-order optimality conditions, that is

$$J_i(d_i(t,y)) = V(t,y) - V(t,y-i), \quad i = 1, \ldots, y.$$  \hfill (12)

The corresponding optimal prices are given by $p_i(t,y) = \pi_i^{-1}(d_i(t,y))$.

The managerial significance of Proposition 1 is that the optimal dynamic price for each batch size has the same property as in the single unit demand case; that is, the marginal revenue of the product is equal to the marginal opportunity cost. This characterization also links decision variables for batches of different size:

**Corollary 1.** For all $t \geq 0$, $y \geq 2$, $i = 2, \ldots, y$, and $i_1 = 1, \ldots, i - 1$ policy variables satisfy

$$J_i(p_i(t,y)) = J_{i_1}(p_{i_1}(t,y)) + J_{i-i_1}(p_{i-i_1}(t,y-i_1)).$$  \hfill (13)

In the next subsection, the additivity property (13) for marginal revenues corresponding to optimal batch prices helps to establish monotonicity relations between these prices.

### 4.2. Relations between prices of batches and those of individual items.

We now apply equation (13) $k$ times “breaking up” the batch. If we let the inventory levels considered in this break up be $y - i \equiv y_{k+1} < y_k < \ldots < y_1 < y_0 \equiv y$ and their differences be $i_j = y_j - y_{j+1}$, $j = 0, \ldots, k$, then we obtain the relation

$$J_i(p_i(t,y)) = \sum_{j=0}^{k} J_{i_j}(p_{i_j}(t,y_j)).$$
Using (11), this relation can be further expanded to

\[ p_i(t, y) - (h_i(p_i(t, y)))^{-1} = \sum_{j=0}^{k} (p_{ij}(t, y_j) - (h_{ij}(p_{ij}(t, y_j)))^{-1}). \]

(14)

With \( k = i - 1 \), the batch is broken up into individual items resulting in

\[ p_i(t, y) - (h_i(p_i(t, y)))^{-1} = \sum_{j=0}^{i-1} (p_1(t, y - j) - (h_1(p_1(t, y - j)))^{-1}). \]

(15)

Compared to \( p_i(t, y) \) (batch price), one can think of \( p_{ij}(t, y_j) \) as prices that would be paid by customers who purchased smaller batches of sizes \( i_j \) adding up to \( i \) in rapid succession. If the batch price \( p_i(t, y) \) is above (below) the total price of sub-batches \( \sum_{j=0}^{k} p_{ij}(t, y_j) \), then the optimal pricing policy exhibits actual quantity premiums (discounts). Note that such premiums or discounts are different from the perceived ones (discussed later in §4.4) resulting from the comparison of \( \frac{p_i(t, y)}{i} \) and \( \frac{p_i(t, y)}{i_j} \). The difference is that only the former can potentially be realized by a customer who needs to purchase a fixed quantity of \( i \) items. Equation (14) suggests that monotonicity relations between batch price \( p_i(t, y) \) and the total price of sub-batches \( \sum_{j=0}^{k} p_{ij}(t, y_j) \) do exist. These relations are described in the following proposition.

**Proposition 2.** Under Assumptions 1-4, the following implications hold:

(a) If \( W_i^u \leq h_i W_i^u \) and \( (h_1(p_i))^{-1} \) is convex, then \( p_i(t, y) \leq \sum_{j=0}^{i-1} p_1(t, y - j) \).

(b) For \( k, i_j \) as defined above, if \( W_i^u \leq h_i W_i^u \) for all \( j = 1, \ldots, k \) and \( (h_i(p_i))^{-1} \) is convex, then \( p_i(t, y) \leq \sum_{j=0}^{k} p_{ij}(t, y_j) \).

(c) If \( W_i^u \geq h_i W_i^u \) and \( (h_1(p_i))^{-1} \) is concave, then \( p_i(t, y) \geq \sum_{j=0}^{i-1} p_1(t, y - j) \).

(d) For \( k, i_j \) as defined above, if \( W_i^u \geq h_i W_i^u \) for all \( j = 1, \ldots, k \) and \( (h_i(p_i))^{-1} \) is concave, then \( p_i(t, y) \geq \sum_{j=0}^{k} p_{ij}(t, y_j) \).

**Remark 1.** The statement may be somewhat refined by an observation that \( (h_1(p_i))^{-1} \) or \( (h_i(p_i))^{-1} \) do not need to be concave or convex for all nonnegative prices. In particular, we only need concavity/convexity of \( (h_1(p_i))^{-1} \) for \( p_i \geq p_1^* \) since optimality conditions imply that \( p_1(t, y) \geq p_1^* \) for all \( t \) and \( y \).

Parts (a) and (b) of Proposition 2 describe the sufficient conditions under which the optimal policy exhibits actual quantity discounts. Parts (c) and (d) describe sufficient conditions for quantity premiums. To better understand these conditions we discuss several examples.

A ready illustration for Proposition 2 is the case of exponential WTP distribution that has a constant hazard rate (consequently, the reciprocal hazard rate is increasing and convex as well as concave). The hazard rate \( h_i(p_i) \) is equal to \( \frac{1}{\mu_i} \) where \( \mu_i \) is the mean of \( W_i^b \). Condition \( W_i^u \leq h_i (\geq h_i) W_i^u \) is equivalent to \( \mu_i \leq (\geq) i \mu_1 \). From parts (a) and (c) of Proposition 2, we have \( p_i(t, y) \leq (\geq) \sum_{j=0}^{i-1} p_1(t, y - j) \) if and only if \( \mu_i \leq (\geq) i \mu_1 \). Thus, the optimal pricing policy will result in a quantity premium (compared to individual items purchased in rapid succession) when \( \mu_i \geq i \mu_1 \) and in a quantity discount when \( \mu_i \leq i \mu_1 \). Effectively, Proposition 2 provides necessary and sufficient conditions for the case of exponential WTP distribution.
Exponential distribution may be the only one with the globally concave reciprocal hazard rate and the support set of all nonnegative numbers. On the other hand, there are several widely used distributions for which the reciprocal of the hazard rate is convex. One example is logistic distribution with \( \pi(p) = \frac{1}{1+\exp\left(-\frac{p}{\sigma}\right)} \) where \( \mu \) and \( \sigma \) are location and scale parameters. The density function is \( f(p) = \frac{\sigma}{\sigma+\exp\left(-\frac{p}{\sigma}\right)} \), therefore \( (h(p))^{-1} = \sigma \left(1 + \exp\left(-\frac{p}{\sigma}\right)\right)\), a decreasing convex function. (While the support set of logistic distribution is all real numbers, the distribution can be truncated without affecting any of the relevant properties.) Another example of decreasing convex \( h(p) \) is given by a normal distribution. (We could not find a reference for this result. Since it may be new, the proof is presented in Appendix A.) A combination of such common distributions with stochastically decreasing \( W_t^u \) (in hazard rate order) guarantees a discount when buying a batch as a whole compared to breaking up this batch into a rapid sequence of smaller purchases.

While conditions of parts (a) and (b) appear as more likely to hold in practice, we point out that they are not “only if” (necessary) conditions. In other words, even though conditions of parts (c) and (d) do not hold, it is still possible that (a) and (b) do not hold either and the optimal premiums are possible only if unit WTP is stochastically increasing in batch size.

Stochastic order conditions in (c) and (d) also lead to the following question with practical implications: is it even possible that unit WTP for some batch size is stochastically smaller than for a batch of a larger size? In the following example, we show that any stochastic order relation between the unit WTP is possible regardless of the unit WTP relations in the customer population prior to realization of the batch size. Consider customer types described by \( \Theta = \mathbb{R}_+ \) (the set of nonnegative real numbers). Let \( \alpha_i \) be given constants describing the order of WTP per unit. As long as \( \frac{\alpha_i}{1} \geq \frac{\alpha_{i+1}}{1+1} \), \( i = 1, \ldots, y \). As long as \( \alpha_i \geq \frac{\alpha_{i+1}}{1+1} \), \( i = 1, \ldots, y - 1 \), we can say that each customer expects quantity discounts since his/her WTP per unit is decreasing. Let customer type distribution be exponential with mean \( \gamma_i \), that is \( \phi(\theta) = \frac{1}{\gamma_i} e^{-\frac{\theta}{\gamma_i}} \), and the segmentation mechanism be such that \( \psi_i(\theta) = B_i e^{-\beta_i \theta} \), where \( B_i, \beta_i > 0 \) for all \( i \) and \( \sum_{i=1}^{y} B_i \leq 1 \). In this case, the resulting fractions of the populations requiring size \( i \) are \( \Psi_i = \frac{B_i}{\gamma_i \beta_i + 1} \), and batch WTP survival functions are \( \pi_i(p) = e^{-\frac{p}{\mu_i}} \) with \( \mu_i = \frac{\alpha_i}{\gamma_i \beta_i + 1} \). That is, batch and unit WTP distributions are exponential with means \( \mu_i \) and \( \mu_i/i \), respectively. Stochastic order between exponential random variables is completely determined by the order between their means. However, the order between \( \mu_i/i \)’s and \( \mu_i/i \)’s in this example can be arbitrary, depending on the values of the segmentation mechanism parameters \( \beta_i \). Moreover, given \( \beta_i \)’s, any ordering of arrival rates \( \lambda_i = \lambda \Psi_i \) can be guaranteed by the appropriate choice of \( B_i \)’s. To summarize, the segmentation mechanism has a critical influence on the inputs of the DNP problem and can result in any WTP stochastic order and arrival rate order structure.

4.3. **Monotonic properties of prices for batches of the same size.** The characterization of the optimal policy provided by Proposition 1 includes the optimality condition for size-one batch as a special case: \( J_1(d_1(t, y)) = V(t, y) - V(t, y - 1) \). Since \( J_1(d_1) \) is decreasing in \( d_i \), \( d_1(t, y) \) is increasing in \( y \) if and only if \( V(t, y) - V(t, y - 1) \) is decreasing in \( y \). The latter condition is equivalent to the concavity of \( V(t, y) \) — the property established by Gallego and van Ryzin (1994) for a single-unit dynamic pricing model. It is also immediate to establish that concavity
of $V(t, y)$ implies that $d_i(t, y)$ are increasing in $y$ for each $i$. However, is $V(t, y)$ concave in the general case? The following proposition provides a condition under which concavity does not hold even for $t$ near 0.

**Proposition 3.** Let $y \geq 1$ be such that $\lambda_y r_y(d_y^*) < \lambda_{y+1} r_{y+1}(d_{y+1}^*)$ then for all sufficiently small $t > 0$ we have $V(t, y) - V(t, y - 1) < V(t, y + 1) - V(t, y)$.

Thus, in general, we cannot expect $V(t, y)$ to be concave in $y$. One special case where concavity holds is discussed next.

Consider a general profit maximization problem corresponding to the customer population with WTP distribution $W_t^b$ and cost $x \geq 0$. Let $\rho_i(x)$ be its optimal value

$$
\rho_i(x) = \max_{d \in [0, 1]} d \pi_i^{-1}(d) - xd
$$

(16)

and $\delta_i(x)$ be its optimal solution (properties of $\rho_i(x)$ and $\delta_i(x)$ are discussed in Lemma 4 of Appendix A). Using $\rho_i(x)$ functions and Proposition 1, the right-hand-side of (7) can be represented as

$$
\sum_{i=1}^{y} \lambda_i \left[ r_i(\delta_i(V(t, y) - V(t, y - i))) - (V(t, y) - V(t, y - i))\delta_i(V(t, y) - V(t, y - i)) \right]
$$

$$
= \sum_{i=1}^{y} \lambda_i \rho_i(V(t, y) - V(t, y - i)).
$$

(17)

This permits to formulate sufficient conditions for the concavity of the value function in terms of the properties of $\rho_i(x)$’s:

**Proposition 4.** Suppose Assumptions 1-4 hold. If $\lambda_i = 0$ for $i \geq 3$, and for all $0 \leq x_1 < x_0 < \infty$ the following conditions hold

$$
\lambda_2[\rho_2(2x_1) - 2\rho_2(x_1 + x_0) + \rho_2(2x_0)] < \lambda_1[\rho_1(x_1) - \rho_1(x_0)],
$$

(18)

$$
\lambda_2 \rho_2(2x_1) < \lambda_1 \rho_1(x_1),
$$

(19)

then $V(t, y)$ is concave in $y$ and $d_i(t, y), i = 1, 2$ is increasing in $y$ for all $t \geq 0$. These properties hold in the strict sense for $t > 0$.

Conditions (18)-(19) are quite general but may be difficult to test directly. The following immediate observation provides a sufficient condition which is easier to test:

**Lemma 3.** Suppose that the following ratio

$$
\frac{\rho_2(2x_1) - 2\rho_2(x_1 + x_0) + \rho_2(2x_0)}{\rho_1(x_1) - \rho_1(x_0)}
$$

(20)

has a finite upper bound $B$ over $0 \leq x_1 < x_2$. If $B \lambda_2 < \lambda_1$ then conditions (18)-(19) hold.

We apply this observation to the exponential distribution as an illustration. Suppose $W_t^b$ and $W_{2}^b$ have means that satisfy $\mu_1 \leq \mu_2 \leq 2\mu_1$. The latter condition is equivalent to $W_t^b \leq_{b} W_2^b$ and $W_1^u \geq_{b} W_2^u$ for the case of exponential distribution. In the Appendix, we show that the ratio (20) has a finite upper bound $\frac{\mu_2}{\mu_1} \leq 2$. According to Lemma 3, as long as $2\lambda_2 < \lambda_1$, the value function is concave.
Relations between unit prices for batches of different size. Recall that break-up monotonicity properties discussed in §4.2 depend only on the characteristics of the hazard rates for unit WTP distributions $W^u$. On the other hand, same-size monotonicity properties depend on the concavity properties of the value function $V(t, y)$ with respect to $y$. Relations between unit prices discussed in this section depend on both. We use the following characterization of concavity: $V(t, y)$ is concave (convex) in $y$ if and only if

$$\frac{V(t, y) - V(t, y - i)}{i} \leq \frac{V(t, y) - V(t, y - i')}{i'}$$

for all $0 < i < i' \leq y$. We also introduce the optimal unit price for the batch of size $i$ as $p^u_i(t, y) = \frac{v_i(t, y)}{i}$. The relation between unit prices becomes transparent if we restate the optimal policy characterization (12) as

$$p^u_i(t, y) - (ih_i(ip^u_i(t, y)))^{-1} = \frac{V(t, y) - V(t, y - i)}{i}.$$ (21)

Concavity (or convexity) of $V(t, y)$ defines a particular ordering of the expressions on the left of (21) for different batch sizes $i$. Given appropriate stochastic ordering of $W^u_i$'s we obtain the following proposition:

**Proposition 5.** If $V(t, y)$ is concave in $y$ and, for some $i < i'$, $W^u_i \leq_{hr} W^u_{i'}$ then $p^u_i(t, y) \leq p^u_{i'}(t, y)$. On the other hand, if $V(t, y)$ is convex in $y$ and, for some $i < i'$, $W^u_i \leq_{hr} W^u_{i'}$ then $p^u_i(t, y) \leq p^u_{i'}(t, y)$.

The proposition provides sufficient conditions for order relations between unit prices. The first part says that if opportunity costs are decreasing (that is, the value function is concave) and customers in different size segments are generally willing to pay more per unit as the batch size increases, then unit prices are increasing in the batch size. On the other hand, if opportunity costs are increasing and customers are willing to pay less as the batch size increases, then unit prices are decreasing in batch size. Since the two conditions do not fully complement each other, there may be intermediate situations in which neither of these conditions hold. Moreover, our discussion of Proposition 4 and Lemma 3 (sufficient conditions for concavity of $V(t, y)$ in $y$) shows that stochastically decreasing (in batch size) unit WTP is more conductive for concavity of the value function. However, in Proposition 5 this stochastic order condition is combined with convexity of $V(t, y)$ in $y$. This indicates complex interplay between concavity/convexity properties of $V(t, y)$ and stochastic order properties of the WTP distributions. We may conjecture that the strength of each particular stochastic order relation between unit WTP (increasing or decreasing) has two opposite influences on the relation between unit prices, and resort to numerical experiments in §5 to refine our insights.

5. Numerical Experiments

In this section, we present numerical experiments which (I) refine the insights about the structure of the optimal DNP policy and (II) examine the benefits of explicit segmentation of customers by batch size. Goal (I) is addressed in a simplified setting where we can maximally leverage analytic results presented in earlier sections. The metric is a quantity premium or discount related to a batch purchase as a fraction of the size-one batch price. The experiment shows complex interplay between the remaining capacity level, concavity properties of the value function, and stochastic order of WTP distributions. We experimentally confirm the intuitive conjecture of §4.4 that this complexity arises because stochastic order of unit WTP has two
opposing influences on unit prices: directly through the optimality conditions and indirectly through the convexity properties of the value function.

The benchmark for (II) is a plausible dynamic pricing heuristic based on single-item dynamic pricing. The benefits are measured as a percentage improvement in the expected revenues of explicit DNP policy over the heuristic one. We also examine structural properties of the DNP policy in the settings used for evaluation of the benefits of DNP.

5.1. Relations between unit prices for batches of different size. We conduct the first experiment in the simplest setting where it is possible to study the influence of two main input components: arrival rates of different size segments and WTP distributions. In particular, the smallest maximal batch size to study nonlinear pricing is two. Also, we use exponential WTP distribution since it permits a complete characterization of monotonicity of prices with respect to batch break up into individual items (according to the discussion in §4.2). As shown in §4.3, exponential distribution also permits a sharper description of sufficient conditions for the concavity of the value function. This significantly simplifies interpretation of the observed behavior of the optimal prices.

We examine a range of arrival rates such that there are 40 expected arrivals during the entire time horizon, and these arrivals are allocated between demand requests for individual items $t\lambda_1$ and size-two batches $t\lambda_2$. Ratio $\frac{\lambda_2}{\lambda_1}$ is varied in the range $[0, 2]$ with 0 corresponding to no size-two arrivals and 2 corresponding to a situation where size-two demand dominates. The ratio of the means $\frac{E_2}{\mu}$ of batch WTP random variables $W_2^b$ and $W_1^b$ varies within the range $[1, 3]$. The lower bound of 1 is the smallest under which we have $W_1^b \leq_{hr} W_2^b$ (Assumption 4). Stochastic order relation $W_2^a \leq_{hr} (\geq_{hr}) W_1^a$ holds for $\frac{E_2}{\mu} \leq (\geq) 2$.

The first insight from this experiment is the range of inputs for which the value function $V(t, y)$ is concave in $y$. We recall that the concavity of $V(t, y)$ is equivalent to increasing optimal price $p_1(t, y)$. Thus, $V(t, y)$ is concave in $y$ (in the local sense, that is, $V(t, y) - V(t, y - 1) \leq V(t, y - 1) - V(t, y - 2)$) if and only if the relative change $\frac{p_1(t, y) - p_1(t, y - 1)}{p_1(t, y)}$ in the price for one unit $p_1(t, y)$ following a one-item sale is positive. We present a contour map of $\frac{p_1(t, y) - p_1(t, y - 1)}{p_1(t, y)}$ as a function of $\frac{E_2}{\mu}$ and $\frac{\lambda_2}{\lambda_1}$ for inventory levels $y \in \{2, 3, 4, 5, 10, 11\}$ in Figure 1. In all six plots presented in this figure, we use the same contour levels $[-0.2, -0.15, \ldots, 0.25, 0.3]$. In the region of inputs $1 \leq \frac{E_2}{\mu} \leq 2$, $0 \leq \frac{\lambda_2}{\lambda_1} \leq \frac{1}{2}$ the ratio $\frac{p_1(t, y) - p_1(t, y - 1)}{p_1(t, y)}$ is positive which is consistent with the sufficient conditions for concavity. We also see that the region of concavity generally corresponds to small values of $\frac{E_2}{\mu}$ and $\frac{\lambda_2}{\lambda_1}$ and grows as $y$ increases. However, this growth is not monotone since the concavity region covers the entire experimental range of inputs whenever $y$ is odd. Thus, the shape of the level curves and the concavity region is drastically different for odd and even levels of inventory. This is due to the presence of the boundary effect that becomes stronger when size-two demand starts to dominate either in terms of the arrival rate or in terms of the unit WTP. Nevertheless, the absolute value of variation in the relative price change shrinks as inventory increases and the value function becomes concave for fixed input values as $y$ increases.

Based on these observations, we hypothesize that when there is no batch demand beyond a certain fixed size, and both the time horizon and the inventory level are sufficiently high, the value function is concave. An intuitive reasoning behind this hypothesis is that the concavity of the value function may be prevented either by the increasing demand for larger batch sizes as the available inventory increases or by the presence of the boundary effects. The first issue
is removed by limiting the maximal batch size. The second issue is removed by sufficiently high values of the available time and inventory.

As we see next, the inventory level also affects the perceived quantity discounts/premiums (through concavity properties of $V(t, y)$). Figure 2 shows contour maps of the perceived relative quantity discount (when negative) or premium (when positive) $\frac{p_2(t, y) - 2p_1(t, y)}{2p_1(t, y)}$ for inventory levels $y \in \{2, 3, 4, 5, 10, 11\}$. The level curves are at $\{-0.2, -0.1, 0.0, 0.1, 0.2, 0.3\}$. The first observation is that, across all inventory levels, a sufficiently small value of $\frac{\mu_2}{\mu_1}$ implies a quantity discount. On one hand, this is natural, since unit WTP for a size-two batch in this case is much smaller (stochastically) than WTP for a size-one batch. On the other hand, the value function for larger inventory levels becomes concave. Thus, a direct price-reducing effect of the small size-two unit WTP dominates the indirect price-increasing effect that the small unit WTP induces via the concavity of the value function.

The second observation is that the minimum level of $\frac{\mu_2}{\mu_1}$ for which the perceived quantity discount occurs for all $\frac{\mu_2}{\mu_1}$ increases in inventory and approaches 2. Recall that $\frac{\mu_2}{\mu_1} = 2$ is the boundary that separates quantity discounts from quantity premiums for another kind of monotonic relation – with respect to the size-two batch breakup into a sequence of two size-one purchases in rapid succession. [This boundary is clearly seen in Figure 3 where we present contour maps of the actual quantity discounts/premiums expressed as the ratio $\frac{p_2(t, y) - (p_1(t, y) + p_1(t, y - 1))}{p_1(t, y) + p_1(t, y - 1)}$.] This is natural since the value function is “flatter” for larger inventory levels (that is, $V(t, y) - V(t, y - 1)$ becomes smaller). Thus, the indirect effect via concavity is more dominated by the direct price-reducing effect of the small size-two unit WTP when the inventory level increases.

The third observation is that for large $\frac{\mu_2}{\mu_1}$ the presence of a discount or a premium and its magnitude is quite sensitive to the inventory level (especially when it is small). For small even $y$, the region of inputs where a discount is observed includes large $\frac{\mu_2}{\mu_1}$ and $\frac{\mu_2}{\mu_1}$. That is precisely the region in which, according to Figure 1, the value function is convex for small even $y$. This means that the indirect price-reducing effect of high unit WTP for size-two batch (via convexity) dominates the direct price-increasing effect of high WTP. This is somewhat surprising but not counterintuitive. The observed range of discounts/premiums is up to 30% which is in line with the WestJet example presented in the introduction.

These observations regarding the structure of the optimal policy indicate that RM systems need to take into account the explicit segmentation of customers according to the required batch size. We evaluate the impact of the explicit DNP policy next.

5.2. Heuristic. Any pricing policy, including the heuristic, needs to take into account two practical issues: 1) it has to respond to booking requests for batches by quoting batch prices, and 2) demand for batches larger than the available capacity is censored. The second issue results in capacity-dependent effective demand characteristics. In particular, the effective demand rate is

$$\tilde{\lambda}(y) = \sum_{i=1}^{g} i\lambda_i, \quad (22)$$

and the effective unit WTP distribution is

$$\tilde{\pi}(p, y) = \left(\sum_{i=1}^{g} i\lambda_i \pi_i(p)\right) / \tilde{\lambda}(y). \quad (23)$$

These inputs permit computation of a single-item dynamic pricing policy by means of the HJB equations

$$\frac{\partial}{\partial t} \hat{V}(t, y) = \max_{p \geq 0} \left\{ \tilde{\lambda}(y) \tilde{\pi}(p, y) [p + \hat{V}(t, y - 1) - \hat{V}(t, y)] \right\}, \quad (24)$$
with the boundary conditions

\[ \tilde{V}(0, y) = 0, \quad \forall y \geq 1, \quad (25) \]
\[ \tilde{V}(t, 0) = 0, \quad \forall t \geq 0. \quad (26) \]

These equations produce single-item dynamic pricing policy \( \tilde{p}(t, y) \) as the price that delivers the maximum in (24). The heuristic multi-item dynamic pricing policy is

\[ p^h_i(t, y) = \sum_{j=0}^{i-1} \tilde{p}(t, y-j). \quad (27) \]

To evaluate the benefits, we use the expected revenue produced by the policy (27).

5.3. **Setup.** The setup of the experiments is based on the exponential valuation distribution. There are up to \( y = 20 \) items in the initial inventory. We examine three patterns of change in WTP distributions so that unit WTP is:

1. increasing in \( i \) by 5\% per item, i.e., the mean of \( W^h_i \) is \( \mu_i = i(1.05)^i \);
2. decreasing in \( i \) by 5\% per item, i.e., the mean of \( W^b_i \) is \( \mu_i = i(0.95)^i \);
3. constant in \( i \), i.e. \( \mu_i = i \).

The first pattern corresponds to a market in which customers are used to airline RM-style pricing and expect to pay a bit more for each additional item. The second pattern corresponds to a market where customers expect quantity discounts. Finally, in the third case, customers expect to pay the same for each item.

For the arrival rates, we examine three patterns (in the order of the decreasing intensity of demand for large batches):

1. “constant” pattern, i.e., \( \lambda_i \) is constant;
2. “slowly decreasing” pattern, i.e., \( \lambda_i \propto i^{-1} \);
3. “quickly decreasing” pattern, i.e., \( \lambda_i \propto i^{-2} \).

To provide a fair comparison, all patterns are normalized to have the same expected volume of unconstrained demand \( t\tilde{\lambda}(y) \) which we refer to as load. Two load levels are examined: a “low” load of 40, and a “high” load of 80. [The numerical implementation solves discrete-time versions of dynamic programming equations with 1000 discretization time periods. Per-period arrival rates are appropriately scaled.]

5.4. **Benefits.** The benefits of DNP for the increasing, decreasing and constant unit WTP patterns are given in Figures 4-6. Larger benefits are generally obtained for the increasing WTP pattern which is somewhat surprising since the heuristic uses prices at consecutive decreasing inventory levels to price batch demand. The benefits are highest (up to 13\% of revenue) for the high load and constant (across different size) arrival rate pattern. The benefits are lower but still significant (up to 1.8\% of revenue) for the decreasing WTP pattern. Finally, the lowest benefits are observed for the “constant” WTP pattern. They are above zero except for the slowly decreasing arrival rate pattern in which the heuristic is optimal.

5.5. **Structure of DNP policy.** We examine representative settings where the benefits are quite high. The first one (Scenario A) corresponds to the high load, constant arrival rates, and *increasing* WTP. The second (Scenario B) corresponds to the high load, constant arrival rates, and *decreasing* WTP. Scenario A (B) falls under cases (c) and (d) (cases (a) and (b)) of Proposition 2 and exhibits increasing (decreasing) monotonicity of prices with respect to batch “break up”. Batch prices \( p_i(t, y) \) in these two scenarios are presented in Figures 7 and 8. Each graph corresponds to \( p_i(t, y) \) for a given \( y \) as a function of batch size \( i \), and shorter graphs
correspond to smaller $y$'s. Both figures do not possess the usual monotonicity patterns. In particular, in Scenario A, the price for each batch size is increasing in the capacity. This is explained primarily by the increase in the effective demand rate for larger capacity levels. In Scenario B, such inverted monotonicity is not observed, but the overlap in the graphs shows that the prices are still not monotonic in the usual sense.

We also examine unit prices $\frac{1}{t} p_i(t, y)$ for given size $i$ as functions of capacity $y$ in Figures 9 and 10. Shorter graphs correspond to larger batches. Unit price is increasing in capacity for Scenario A, but not in Scenario B. This is natural since unit WTP is increasing only in Scenario A. However, in both scenarios, unit price is decreasing in batch size (shorter graphs are lower). This is somewhat surprising in Scenario A since customers are prepared to pay more for larger batches.

6. Concluding Remarks

Although our work focuses on the analysis and insights for a general problem of dynamic nonlinear pricing, the proposed model and its analysis can be extended to broader settings. In the current model, we consider a situation when a customer requires a fixed amount of product, i.e., he/she does not face a choice between batches of different size. What if there is a choice? A natural extension of the model would consider a setting when the market segmentation between batches of different size is imperfect, and each customer makes a choice between batches of different size. In the short term, customers may also behave strategically trying to fill the desired order through several purchases of smaller orders made at different times. This results in a dynamic pricing problem with batched demand in the presence of consumer choice. Finally, in a dynamic pricing problem on a network, break up monotonicity results hold under certain conditions which may give an opportunity to better understand the structure of this important and complex problem.

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References

Appendix A. Mathematical Proofs

\((h(p))^{-1}\) convexity proof for the case of Normal distribution. In general, the first two derivatives of \((h(p))^{-1}\) are

\[
\frac{d}{dp}(h(p))^{-1} = \frac{d}{dp} \left( \frac{\pi(p)}{f(p)} \right) = -1 - \frac{\pi(p) f'(p)}{(f(p))^2} = -1 - \frac{\pi(p)}{f(p)} \frac{d}{dp}(\ln f(p)),
\]

\[
\frac{d^2}{dp^2}(h(p))^{-1} = - \frac{d}{dp} \left( \frac{\pi(p)}{f(p)} \right) \frac{d}{dp}(\ln f(p)) - \frac{\pi(p)}{f(p)} \frac{d^2}{dp^2}(\ln f(p))
\]

\[
= \left( 1 + \frac{\pi(p)}{f(p)} \frac{d}{dp}(\ln f(p)) \right) \frac{d}{dp}(\ln f(p)) - \frac{\pi(p)}{f(p)} \frac{d^2}{dp^2}(\ln f(p))
\]

\[
= \frac{d}{dp}(\ln f(p)) + \frac{\pi(p)}{f(p)} \left( \left( \frac{d}{dp}(\ln f(p)) \right)^2 - \frac{d^2}{dp^2}(\ln f(p)) \right).
\]

For normal distribution with mean \(\mu\) and standard deviation \(\sigma\), we have \(\ln f(p) = -\frac{(p-\mu)^2}{2\sigma^2} + C\), where \(C\) is a constant. Therefore, \(\frac{d}{dp}(\ln f(p)) = -\frac{p-\mu}{\sigma^2}\) and \(\frac{d^2}{dp^2}(\ln f(p)) = -\frac{1}{\sigma^2}\). For \(p \leq \mu\), all terms in the expression for \(\frac{d^2}{dp^2}(h(p))^{-1}\) are nonnegative. Consider \(p > \mu\). Inequality 7.1.13 of Abramowitz and Stegun (1972) states that, for \(x \geq 0\),

\[
\frac{1}{e^{-x^2}} \int_x^{+\infty} e^{-t^2} dt > \frac{1}{x + \sqrt{x^2 + 2}}.
\]

Since, for general normal distribution,

\[
\frac{\pi(p)}{f(p)} = \sqrt{2\sigma} \frac{1}{e^{-\left(\frac{p-\mu}{\sqrt{2}\sigma}\right)^2}} \int_{\frac{p-\mu}{\sqrt{2}\sigma}}^{+\infty} e^{-t^2} dt,
\]

we let \(x = \frac{p-\mu}{\sqrt{2}\sigma}\) and obtain

\[
\frac{\pi(p)}{f(p)} > \frac{\sqrt{2}\sigma}{\frac{p-\mu}{\sqrt{2}\sigma} + \sqrt{\left(\frac{p-\mu}{\sqrt{2}\sigma}\right)^2 + 2}} = \frac{2\sigma}{\frac{p-\mu}{\sigma} + \sqrt{\left(\frac{p-\mu}{\sigma}\right)^2 + 4}}.
\]
It follows that \( \frac{d^2}{dp^2}(h(p))^{-1} > \frac{1}{\sigma} K \left( \frac{p - \mu}{\sigma} \right) \) where
\[
K(z) = -z + \frac{2(z^2 + 1)}{z + \sqrt{z^2 + 4}} = \frac{z^2 + 2 - z\sqrt{z^2 + 4}}{z + \sqrt{z^2 + 4}}.
\]
We see that \( K(z) > 0 \) for all \( z \geq 0 \), since \((z^2 + 2)^2 = z^4 + 4z^2 + 4 > z^2(z^2 + 4)\).

**Proof of Lemma 1.** Consider inequalities \( \pi_{i+1}^{-1}(d_{i+1}) \geq \pi_i^{-1}(d_i) \) which define \( \mathcal{D}^y \). Each inequality can be equivalently expressed as \( \pi_i(\pi_{i+1}^{-1}(d_{i+1})) - d_i \leq 0 \). Thus, set \( \mathcal{D}^y \) is convex if, for each \( i \), function \( G_i(d) = \pi_i(\pi_{i+1}^{-1}(d)) \) is convex. The latter holds if the derivative
\[
G'_i(d) = \frac{f_i(\pi_{i+1}^{-1}(d))}{f_{i+1}(\pi_{i+1}^{-1}(d))}
\]
is increasing. This is true since the ratio \( \frac{f_i(p)}{f_{i+1}(p)} \) is decreasing in \( p \) by Assumption 4 and \( \pi_{i+1}^{-1}(d) \) is decreasing in \( d \).

**Proof of Proposition 1.** Strict concavity of the objective is immediate from Assumption 3. Convexity of the feasible set follows from Lemma 1. The problem also satisfies Slater’s condition. Thus, the first-order optimality (KKT) conditions are necessary and sufficient. Uniqueness follows from strict concavity of the objective function. It remains to show that the solution to the subproblem in (7) exists and satisfies KKT conditions.

Indeed, \( V(t, y) - V(t, y - i) \geq 0 \), and the range of \( J_i(d) \) includes all nonnegative values (since \( \lim_{d \to 0} J_i(d) = \lim_{p \to -\infty} J_i(p) = \infty \)). Therefore, solution to (12) exists. Moreover, \( p_i(d, y) \leq p_{i+1}(d, y) \) holds because
\[
J_i(p_i(t, y)) = V(t, y) - V(t, y - i) \leq V(t, y) - V(t, y - i - 1) = J_{i+1}(p_{i+1}(t, y)) \leq J_i(p_{i+1}(t, y))
\]
(where the last inequality follows from Lemma 2) and \( J_i(p_i) \) is increasing (as mentioned earlier, this follows from Assumption 3). Thus, solution to (12) is feasible. Finally, (12) represents KKT conditions for the case where all Lagrange multipliers are zero.

**Proof of Proposition 2.** From (14), it follows that \( p_i(t, y) \leq (\geq) \sum_{j=0}^{k} p_{ij}(t, y_j) \) if and only if
\[
(h_i(p_i(t, y)))^{-1} \leq (\geq) \sum_{j=0}^{k} (h_{ij}(p_{ij}(t, y)))^{-1}.
\]

In part (a), suppose that \( p_i(t, y) > \sum_{j=0}^{i-1} p_1(t, y - j) \). Then
\[
(h_i(p_i(t, y)))^{-1} \leq (\geq) \sum_{j=0}^{i-1} (h_{ij}(p_{ij}(t, y)))^{-1}.
\]
In part (b), suppose that \( p_i(t,y) > \sum_{j=0}^{k} p_{ij}(t,y) \) and observe that \( \sum_{j=0}^{k} \frac{i_j}{i} = 1 \). By applying convexity property of \( (h_i(p_i))^{-1} \) before stochastic order conditions \( W^u_i \leq hr W^u_{ij} \), we obtain
\[
(h_i(p_i(t,y)))^{-1} \leq \left( h_i \left( \sum_{j=0}^{k} \frac{i_j}{i} p_{ij}(t,y) \right) \right)^{-1} = \left( \sum_{j=0}^{k} \frac{i_j}{i} h_i(p_{ij}(t,y)) \right)^{-1} \leq \sum_{j=0}^{k} \frac{i_j}{i} (h_i(p_{ij}(t,y)))^{-1} = \sum_{j=0}^{k} (h_i(p_{ij}(t,y)))^{-1},
\]
again, a contradiction.

Proofs of parts (c) and (d) are similar to those of (a) and (b), respectively.

**Proof of Proposition 3.** Since \( V(0,y) = 0 \), for all \( y \geq 0 \), observe from (7) that \( \frac{\partial}{\partial t} V(t,y)|_{t=0} = \sum_{i=1}^{y} \lambda_i r_i(d_i^*) \). Therefore, \( \frac{\partial}{\partial t}[V(t,y) - V(t,y - 1)]_{t=0} = \lambda_y r_y(d_y^*) < \lambda_{y+1} r_{y+1}(d_{y+1}^*) = \frac{\partial}{\partial t}[V(t,y + 1) - V(t,y)]_{t=0} \) Function \( V(t,\cdot) \) is piecewise continuously differentiable in \( t \) as the solution to (7). Therefore, this strict inequality between derivatives will hold in some small neighborhood of \( t = 0 \). The claim of the proposition follows.

**Proof of Proposition 4.** We begin the proof by a brief discussion of the problem (16) and its optimal value \( \rho_i(x) \) and optimal solution \( \delta_i(x) \). The following observations are immediate:

**Lemma 4.** If Assumptions 1-3 hold, then
(a) For any \( x \geq 0 \), there exists the unique optimal solution \( \delta_i(x) \in (0,1) \) given by equation \( J_i(\delta_i(x)) = x \).
(b) We have \( \delta_i'(x) = \frac{1}{J_i(\delta_i(x))} \) and \( \delta_i''(x) = -\frac{J_i''(\delta_i(x))}{(J_i(\delta_i(x)))^3} \) for any \( x \geq 0 \). Therefore, \( \delta_i(x) \) is a strictly decreasing function of \( x \). It is convex in \( x \) if and only if \( J_i(x) \) is convex.
(c) We have \( \rho_i'(x) = -\delta_i(x) \) and \( \rho_i''(x) = -\delta_i'(x) \) for any \( x \geq 0 \). Therefore, \( \rho_i(x) \) is a strictly decreasing and strictly convex function of \( x \).
(d) \( \lim_{x \to \infty} \delta_i(x) = 0 \) and \( \lim_{x \to \infty} \rho_i(x) = 0 \).

We can now proceed with the proof which is by induction on the inventory level. The base case is \( y = 1 \).

Continuity of arrival rates implies that the value function is continuously differentiable. Consider a set \( T_1 = \{ t \geq 0 : V(t',2) \leq 2V(t',1) \ \forall t' \in [0,t] \} \). For \( \bar{t} = \sup T_1 \) we must have \( \frac{\partial}{\partial t}(V(t,2) - 2V(t,1))|_{t=\bar{t}} \geq 0 \). We demonstrate the opposite, arriving at contradiction. Since \( V(t,1) \) and \( V(t,y) \) are continuous, it should be true that \( V(\bar{t},2) = 2V(\bar{t},1) \). Using representation (17) with \( x = V(\bar{t},1) \) we get
\[
\frac{\partial}{\partial t}V(t,1)|_{t=\bar{t}} = \lambda_{2\bar{t}} \rho_2(2x) + \lambda_{1\bar{t}} \rho_1(x),
\]
\[
\frac{\partial}{\partial t}V(t,1)|_{t=\bar{t}} = \lambda_{1\bar{t}} \rho_1(x).
\]
This implies
\[
\frac{\partial}{\partial t}(V(t,2) - 2V(t,1))|_{t=\bar{t}} = \lambda_{2\bar{t}} \rho_2(x) - \lambda_{1\bar{t}} \rho_1(x) < 0
\]
by assumption (19).

Suppose that concavity holds up to some level of inventory \( y \geq 2 \). We now establish it for \( y + 1 \). Consider a set \( T_y = \{ t \geq 0 : V(t', y + 1) - V(t', y) \leq V(t', y) - V(t', y - 1) \ \forall t' \in [0, t] \} \) and \( \bar{t}_y = \sup T_y \). Let \( x_j = V(\bar{t}_y, j + 1) - V(\bar{t}_y, j) \) for \( j \geq 0 \). For \( j < 0 \) we let \( x_j = \infty \). By the inductive assumption, we have \( x_{j+1} < x_j, \ j \geq 0 \). By definition of \( \bar{t}_y \) and continuity of the value function we have \( x_{y+1} = x_y \). Using representation (17) we get

\[
\frac{\partial}{\partial t} V(t, y + 1)|_{t=\bar{t}_y} = \lambda_2 \rho_2(2x_y) + \lambda_1 \rho_1(x_y),
\]

\[
\frac{\partial}{\partial t} V(t, y)|_{t=\bar{t}_y} = \lambda_2 \rho_2(2x_y + x_{y-1}) + \lambda_1 \rho_1(x_y),
\]

\[
\frac{\partial}{\partial t} V(t, y - 1)|_{t=\bar{t}_y} = \lambda_2 \rho_2(x_{y-1} + x_{y-2}) + \lambda_1 \rho_1(x_{y-1}),
\]

and

\[
\frac{\partial}{\partial t}[V(t, y + 1) - 2V(t, y) + V(t, y - 1)]|_{t=\bar{t}_y} = \lambda_2 \rho_2(2x_y - 2x_y + x_{y-1} + \rho_2(x_{y-1} + x_{y-2})] - \lambda_1 \rho_1(x_y) - \rho_1(x_{y-1}).
\]

Since \( x_{y-2} > x_{y-1} \) we have \( x_{y-1} + x_{y-2} > 2x_{y-1} \) and \( \rho_2(x_{y-1} + x_{y-2}) < \rho_2(2x_{y-1}) \). Condition (18) implies that \( \frac{\partial}{\partial t}[V(t, y + 1) - 2V(t, y) + V(t, y - 1)]|_{t=\bar{t}_y} < 0 \). Again, a contradiction.

Monotonicity of the optimal policy is an immediate consequence of concavity of the value function.

**Illustration to Lemma 3 for the case of exponential WTP distributions.** It is immediate to see that \( \rho_i(x) = \mu_i e^{-x/\mu_i}, i = 1, 2 \). Letting \( x_1 = x \) and \( x_2 = x + \Delta \), where \( x \geq 0 \) and \( \Delta > 0 \) we write the ratio in question as

\[
\frac{\mu_2 [e^{-x/\mu_2} - 2e^{-x+\Delta/\mu_2} + e^{-x+2\Delta/\mu_2}]}{\mu_1 [e^{-x/\mu_1} - e^{-x+\Delta/\mu_1}]} = \frac{\mu_2}{\mu_1} \times \frac{e^{-x/\mu_2}}{e^{-x/\mu_1}} \times \frac{1 - 2e^{-\Delta/\mu_2} + e^{-2\Delta/\mu_2}}{1 - e^{-\Delta/\mu_1}}.
\]

Observe that \( e^{-\Delta/\mu_2} \leq e^{-\Delta/\mu_1} \) for all \( x \geq 0 \) because \( \mu_2 \leq 2\mu_1 \). Moreover,

\[
\frac{1 - 2e^{-\Delta/\mu_2} + e^{-2\Delta/\mu_2}}{1 - e^{-\Delta/\mu_2}} = \frac{(1 - e^{-\Delta/\mu_2})^2}{1 - e^{-\Delta/\mu_1}}
\]
is an increasing function of \( \Delta \) with the limit of 1 as \( \Delta \to \infty \). Indeed, the derivative of this expression is

\[
\frac{d}{d\Delta} \left[ \frac{(1 - e^{-\frac{\Delta}{\mu_2}})^2}{1 - e^{-\frac{\Delta}{\mu_1}}} \right] = \frac{2(1 - e^{-\frac{\Delta}{\mu_2}}) e^{-\frac{\Delta}{\mu_2}} (1 - e^{-\frac{\Delta}{\mu_1}}) - (1 - e^{-\frac{\Delta}{\mu_2}})^2 e^{-\frac{\Delta}{\mu_1}}}{(1 - e^{-\frac{\Delta}{\mu_1}})^2}
\]

\[
= \frac{1 - e^{-\frac{\Delta}{\mu_2}}}{\mu_1 (1 - e^{-\frac{\Delta}{\mu_1}})^2} \left[ 2\mu_2 e^{-\frac{\Delta}{\mu_2}} (1 - e^{-\frac{\Delta}{\mu_1}}) - (1 - e^{-\frac{\Delta}{\mu_2}}) e^{-\frac{\Delta}{\mu_1}} \right]
\]

\[
\geq \frac{1 - e^{-\frac{\Delta}{\mu_2}}}{\mu_1 (1 - e^{-\frac{\Delta}{\mu_1}})^2} \left[ e^{-\frac{\Delta}{\mu_2}} (1 - e^{-\frac{\Delta}{\mu_1}}) - (1 - e^{-\frac{\Delta}{\mu_2}}) e^{-\frac{\Delta}{\mu_1}} \right]
\]

\[
= \frac{1 - e^{-\frac{\Delta}{\mu_2}}}{\mu_1 (1 - e^{-\frac{\Delta}{\mu_1}})^2} \left[ e^{-\frac{\Delta}{\mu_2}} - e^{-\frac{\Delta}{\mu_1}} \right].
\]

The latter is nonnegative since \( \mu_2 \geq \mu_1 \) and, therefore, \( e^{-\frac{\Delta}{\mu_2}} \geq e^{-\frac{\Delta}{\mu_1}} \). Therefore, in the current example, the upper bound for the ratio is equal to \( \frac{\mu_2}{\mu_1} \).

**Proof of Proposition 5.** To prove the first part, observe that by concavity we have

\[
p_i^u(t, y) - (ih_i(ip_i^u(t, y)))^{-1} = \frac{V(t, y) - V(t, y - i)}{i} \leq \frac{V(t, y) - V(t, y - i')}{i'} = p_i^u(t, y) - (i'h_i' (i'p_i^u(t, y)))^{-1}.
\]

Since \( W_i^u \leq W_i \) implies that \( ih_i(ip_i^u(t, y)) \geq i'h_i' (i'p_i^u(t, y)) \), it follows that \( p_i^u(t, y) - (i'h_i' (i'p_i^u(t, y)))^{-1} \leq p_i^u(t, y) - (ih_i(ip_i^u(t, y)))^{-1} \) and

\[
p_i^u(t, y) - (ih_i(ip_i^u(t, y)))^{-1} \leq p_i^u(t, y) - (ih_i(ip_i^u(t, y)))^{-1}.
\]

Since \( w - (ih_i(iw))^{-1} \) is increasing, we have \( p_i^u(t, y) \leq p_i^u(t, y) \). The proof of the second part is identical with the order of the inequalities reversed.

**Proof of Proposition 6.** Regularity Assumptions 1-2 imply that the problem has an optimal solution. The feasible set is convex because of Assumption 4, and the objective is strictly concave because of Assumption 3. As a result, the optimal solution is unique and the first-order (KKT) optimality conditions are necessary and sufficient. Lagrangian function of the problem (28)-(31) has the form

\[
L = \sum_{i=1}^{y} Q_i(t) r_i(d_i) + z \left( y - \sum_{i=1}^{y} Q_i(t) d_i \right) + \sum_{i=1}^{y-1} x_i (\pi_{i+1}^{-1} - \pi_i^{-1}) (d_i) + \sum_{i=1}^{y} [v^0 d_i + v^1 (1 - d_i)]
\]

where \( z, x_i, v^0 \) and \( v_i \) are Lagrange multipliers. KKT conditions include primal feasibility, complimentary slackness, non-negativity of Lagrange multipliers and the fact that the optimal solution is a stationary point of the Lagrangian. The latter condition is expressed as the following system of equations for \( d_i \)’s

\[
\frac{\partial L}{\partial d_i} = Q_i(t) J_i(d_i) - z Q_i(t) + (x_{i-1} - x_i) (\pi_i^{-1})' (d_i) + v^0 - v^1 = 0, \quad i = 1, \ldots, y.
\]

When all Lagrange multipliers except \( z \) are zeros, this system reduces to \( J_i(d_i) = z, \ i = 1, \ldots, y \). For any \( z \geq 0 \), this system has the unique solution \( d_i^z \) which satisfies (30) and (31). This is obvious for (31). Let \( p_i^z = \pi_i^{-1} (d_i^z) \) and observe that \( J_i(p_i^z) = z \). Since \( J_i(p) \geq J_{i+1}(p) \) for all \( p \) (by Lemma 2) and \( J_i(p) \) is increasing, it follows from \( J_i(p_i^z) = J_{i+1}(p_i^z) \) that \( p_i^z \leq p_{i+1}^z \), verifying (30).
Thus, we can assume that all Lagrange multipliers except $z$ are zero. If $\sum_{i=1}^{y} Q_i(t) d_i^* \leq y$, vector $d^*$ satisfies KKT conditions with $z = 0$, and thus, it is optimal. Otherwise, there is strictly positive Lagrange multiplier $z = z(t,y)$ and, by complimentary slackness, constraint (29) is active. Thus, the solution is given by vector $d^0(t,y)$.

**Appendix B. Deterministic Approximation**

Similarly to Gallego and van Ryzin (1997) we base our approximation on the deterministic version of the problem:

$$V^D(t,y) = \sup \sum_{i=1}^{y} \int_{0}^{t} \lambda_i r_i(d_{i\tau}) d\tau$$

subject to $\sum_{i=1}^{y} \int_{0}^{t} \lambda_i d_{i\tau} d\tau \leq y$,

$$\pi_i^{-1}(d_{i\tau}) \leq \pi_{i+1}^{-1}(d_{(i+1)\tau}), \quad i = 1, \ldots, y-1, \quad \tau \in [0,t],$$

$$0 \leq d_{i\tau} \leq 1, \quad i = 1, \ldots, y-1, \quad \tau \in [0,t].$$

Since the WTP distribution does not depend on $\tau$, it is easy to see that the optimal trajectory of $d_{i\tau}$’s does not depend on $\tau$. The problem then takes a static form, where we use $Q_i(t) = t\lambda_i$ to denote cumulative quote request arrivals for batches of size $i$:

$$V^D(t,y) = \max \sum_{i=1}^{y} Q_i(t) r_i(d_i)$$

subject to $\sum_{i=1}^{y} Q_i(t) d_i \leq y$,

$$\pi_i^{-1}(d_i) \leq \pi_{i+1}^{-1}(d_{i+1}), \quad i = 1, \ldots, y-1,$$

$$0 \leq d_i \leq 1, \quad i = 1, \ldots, y-1.$$

A direct application of Theorem 1 of Gallego and van Ryzin (1997) shows that, under regularity Assumptions 1 and 2 $V^D(t,y)$ provides an upper bound for the optimal value of the firm’s problem:

$$V^D(t,y) \geq V(t,y) \quad \text{for all } t \geq 0 \text{ and } y = 1,2,\ldots.$$

To describe the optimal value of the problem (28)-(31), we define the vector $d^0(t,y) = (d^0_i(t,y))_{i=1}^{y}$ as the solution $d$, together with a scalar $z(t,y) = z$, to the following system of equations

$$\sum_{i=1}^{y} Q_i(t) d_i = y,$$

$$J_i(d_i) = z, \quad i = 1,\ldots, y.$$  

We also use values $d_i^*$ defined previously as maximizers of $r_i(d_i)$. As such, these values satisfy first-order optimality conditions $J_i(d_i^*) = 0, \quad i = 1,\ldots, y$.

**Proposition 6.** Under Assumptions 1-4 there exists the unique optimal optimal solution $\bar{d}(t,y)$ to (28)-(31). If $\sum_{i=1}^{y} Q_i(t) d_i^* \leq y$, then $\bar{d}(t,y) = d^*$. Otherwise, $\bar{d}(t,y) = d^0(t,y)$ such that $z(t,y) > 0$. 


Proposition 6 essentially reduces the problem (28)-(31) to solving a collection of single-variable equations of the form (33). Indeed, we can start by testing whether \( \mathbf{d}^* \), which solves (33) with \( z = 0 \), satisfies (29). If it does, the optimal solution is found, otherwise, we can pick a sufficiently high \( z > 0 \) so that (29) is strictly satisfied for the resulting \( \mathbf{d} \) vector and employ bisection search on \( z \) until the solution is found within specified tolerance.

Using price vector \( \bar{\mathbf{p}}(t, y) \) corresponding \( \bar{\mathbf{d}}(t, y) \) we can adopt the Make-to-Order (MTO) heuristic policy of Gallego and van Ryzin (1997) and its performance guarantee in Theorem 3 to the DNP problem. The heuristic takes the following form:

**MTO Policy:** *Price all batches with size \( i \) within the current capacity according to the fixed price \( \bar{p}_i(t, y) \). For all batches in excess of the available capacity set price to \( \infty \).*

The asymptotic performance guarantee applies as long as there is a maximum batch size for each demand rate is non-zero.
Figure 1. Contour maps of the relative change in the price for one unit following a one-item sale as a function of ratios $\frac{\mu_2}{\mu_1}$ and $\lambda_2/\lambda_1$. 

$y = 2$  

$y = 3$  

$y = 4$  

$y = 5$  

$y = 10$  

$y = 11$  

$\frac{\mu_2}{\mu_1}$  

$\frac{\lambda_2}{\lambda_1}$
Figure 2. Contour maps of the perceived quantity discount or premium for a two-unit batch (the fraction of the double price of a one-unit batch) as a function of ratios $\frac{\mu_2}{\mu_1}$ and $\frac{\lambda_2}{\lambda_1}$.
Figure 3. Contour maps of the actual quantity discount or premium for a two-unit batch (the fraction of the prices of two one-unit batches bought in rapid succession) as a function of ratios $\frac{\mu_2}{\mu_1}$ and $\frac{\lambda_2}{\lambda_1}$.
**Figure 4.** Benefits of the DNP policy for increasing WTP pattern

**Figure 5.** Benefits of the DNP policy for decreasing WTP pattern

**Figure 6.** Benefits of the DNP policy for constant WTP pattern

**Figure 7.** Batch price $p_i(t, y)$ for given capacity $y$ as a function of batch size $i$ in Scenario A
**Figure 8.** Batch price $p_i(t, y)$ for given capacity $y$ as a function of batch size $i$ in Scenario B

![Graph of Figure 8](image)

**Figure 9.** Unit price $\frac{1}{i}p_i(t, y)$ for given batch size $i$ as a function of capacity $y$ in Scenario A

![Graph of Figure 9](image)

**Figure 10.** Unit price $\frac{1}{i}p_i(t, y)$ for given batch size $i$ as a function of capacity $y$ in Scenario B

![Graph of Figure 10](image)