Task-Specific Abilities in
Multi-Task Principal-Agent Relationships

Veikko Thiele†

Queen’s School of Business
Queen’s University

Forthcoming in Labour Economics

December 2009

Abstract
This paper analyzes a multi-task agency framework where the agent exhibits task-specific
abilities. It illustrates how incentive contracts account for the agent’s task-specific abilities
if contractible performance measures do not reflect the agent’s multidimensional contribu-
tion to firm value. This paper further sheds light on potential ranking criteria for perform-
ance measures in multi-task principal-agent relationships. It demonstrates that perform-
ance measures in multi-task agencies cannot necessarily be compared by their respective
signal-to-noise ratio as in single-task agency relationships. In fact, it is indispensable to
take the induced effort distortion and the measure-cost efficiency into consideration—both
determined by the agent’s task-specific abilities.

Keywords: Task-specific human capital, performance measurement, distortion, multi-task
agencies, congruence, incentives.

JEL classification: D23, D82, J24

*The author would like to thank Ian Walker (the editor), two anonymous referees, Gerald Feltham, Sandra
Chamberlain, Chia-Chun Hsieh, Anja Schöttner, Louis-Philippe Sirois, and Chee-Wee Tan for valuable comments
and suggestions.
†Address: Goodes Hall, 143 Union Street, Kingston, ON, K7L 3N6, Canada, phone: 1-613-533-2738, fax
1-613-533-6589, e-mail: vthiele@business.queensu.ca.
1 Introduction

Empirical studies have offered an abundance of evidence suggesting that individuals are highly responsive to monetary incentive (see e.g. Asch (1990), Paarsch and Shearer (1999) and Lazear (2000)). Nonetheless, the specific effects of reward schemes are somewhat ambiguous when individuals are required to perform a collection of different tasks. In such situations, Kerr (1975) cautioned against the consequences of adopting reward systems that inefficiently overemphasize some tasks while underemphasizing others. This is because such reward schemes would motivate individuals to focus on less valuable tasks while disregarding the more important ones (Kerr, 1975).

One example for this phenomenon has attracted considerable attention not only from the public media, but also from many scholars: In the last decade, some states in the U.S. began to use the results of standardized school-level tests to reward or sanction individual teachers with the intention to improve the overall teaching quality.\(^1\) For instance, the state California spent almost $700 million for financial incentives in 2001, whereby individual teachers could get up to $25,000 as a bonus in schools with the highest performance improvement (Kane and Staiger, 2002). The motivation for such incentive schemes stems from the (clearly mistaken) belief that the performance of students in these tests adequately reflects teaching quality. Kane and Staiger (2002), however, raised concerns about the reliability of school-level tests as they represent only a small fraction of teaching responsibilities such as teaching basic rather than advanced knowledge; or they tend to focus on particular subjects like Math or English.

It is quite predictable that the provision of monetary incentives based on students’ achievements in school-level tests eventually motivates teachers to shift their teaching emphasis to the knowledge and skills covered by these tests. In fact, Stecher and Barron (1999) provide empirical evidence of teachers in Kentucky spending more time on science in fourth grade when science was tested; and on Math in fifth grade when Math was tested. Other studies provide further evidence that teachers shifted their teaching emphasis to tested, at the expense of non-tested knowledge; see e.g. Klein, Hamilton, McCaffrey, and Stecher (2000) and Jacob (2005).\(^2\)

---

\(^1\)See Kane and Staiger (2002) for a detailed description of school-level tests in the U.S. and their differences among states.

\(^2\)Some teachers also began to manipulate their students’ test scores in order to enhance their own performance evaluations. For example, Jacob and Levitt (2003) estimated that at least four percent of classroom tests in Chicago
Ironically, teachers did what they were rewarded for: improving their students’ test-scores. To achieve this objective, teachers adjusted the allocation of their respective effort to the different tasks—a response to the implemented incentive scheme which was not intended.³

The school-level test example illustrates how incentive schemes based on performance measures, which do not perfectly capture the multidimensional contributions of agents, motivate these agents to implement inefficient effort allocations across the relevant tasks.⁴ Such performance measures are commonly referred to as incongruent performance measures in the multi-task principal-agent literature (see e.g. Feltham and Xie (1994)). The consequences of using incongruent performance measures on the efficiency of incentive contracts have been analyzed e.g. by Feltham and Xie (1994) and Baker (2002). These studies, however, disregard the possibility that agents may perform some tasks more efficiently than others—a fact well known in labour economics.⁵ Recent literature specifically emphasizes the role of acquiring human capital for particular tasks; see e.g. Lindbeck and Snower (2000), Gibbons and Waldman (2006) and Gibbons and Waldman (2004).⁶ Since individuals differ in their learning aptitudes which inevitably lead to discrepancies in their skills and abilities (Gibbons and Waldman, 2006), it is reasonable to conclude that individuals differ with respect to their task-specific productivities.

To understand the nature of incentive contracts in multi-task principal-agent relationships, it is essential to investigate whether and how task-specific abilities of agents affect their respective effort choices, and thus, their optimal inventive contracts. This paper therefore analyzes a multi-task principal-agent relationship in order to gain new insights into the optimal provision of effort incentives when agents possess different abilities for performing the relevant tasks, and available performance measures do not fully reflect the potential contributions of agents to firm value. We derive specific measures to evaluate the efficiency of the agent’s incentive contract, elementary schools were affected by such manipulations. For further examples of encountered manipulations see e.g. Tysome (1994), Hofkins (1995), and Goodnough (1999).

³Maher, Ramanathan, and Peterson (1979) conceived the term ‘congruence of perception with preferences’ to refer to the phenomenon that even if an individual has the correct perception of the relative importance of tasks, there may still be a preference for specific tasks.

⁴For further illustrative examples refer to Kerr (1975), Gibbons (1998), Prendergast (1999), and Baker (2000).

⁵Schnedler (2008) is an exception. However, his focus is different in the sense that he investigates under what circumstances incongruent performance measures can generate a higher surplus than perfectly congruent performance measures.

⁶For empirical evidence see Baker, Gibbs, and Holmström (1994).
and illustrate how these measures are affected by the agent’s task-specific abilities.

This paper further investigates how the value of performance measures can be compared in multi-task agency relations. We show that the signal-to-noise ratio—sufficient to rank performance measures in single-task agency relationships (see e.g. Kim and Suh (1991))—can only be used in multi-task agency relations if all performance measures provide the same information about the agent’s relative effort allocation. For cases where performance measures contain different information about the agent’s relative effort allocation, we derive a more general ranking criterion which is highly sensitive to the agent’s task-specific abilities. Our more general ranking criterion has therefore an important implication: In contrast to single-task agencies, the relative value of specific performance measures in multi-task agency relationships is closely tied to the characteristics of the agent, in particular to his task-specific abilities.

This paper combines two strands of economic literature. First, the analyzed framework in this paper builds on the multi-task agency model developed by Holmström and Milgrom (1991), and includes incongruent performance measures as analyzed by Feltham and Xie (1994), Bushman, Indjejikian, and Penno (2000), Banker and Thevaranjan (2000), Datar, Kulp, and Lambert (2001), and Baker (2002). Second, it incorporates task-specific human capital in the sense of Gibbons and Waldman (1999), Lindbeck and Snower (2000), and Gibbons and Waldman (2004, 2006). In short, the key contribution of this paper to the literature on incentive contracts is the consideration of task-specific abilities in a multitasking framework. It shows that discrepancies in task-specific abilities among agents can explain why they potentially receive different incentive contracts even if their jobs are identical; or why they are assigned to different jobs.

This paper proceeds as follows. We introduce the basic model in Section 2, and derive the first-best solution in Section 3. We then derive in Section 4 the second-best incentive contract, and focus on the relationship between performance measure congruity and effort distortion in Section 5. In Section 6, we pursue the question of how performance measures can be ranked in multi-task agency relationships, in particular when agents exhibit task-specific abilities. We then discuss in Section 7 the main implications for the efficient job design and job assignment within organizations. Section 8 summarizes the key insights and concludes.

7The signal-to-noise ratio relates the information content of a signal to its variance, and thus measures the precision of this signal.
2 The Model

Consider a single-period employment relationship between a risk-neutral principal and a risk-averse agent. The agent is in charge of conducting \( n \geq 2 \) tasks (multitasking), which cannot be split and allocated to different agents.\(^8\) The agent is thus required to implement a vector of effort \( e = (e_1, \ldots, e_n)^T, e \in \mathbb{R}^n^+ \), where \( e_i \) denotes his effort which is allocated to task \( i \).\(^9\) All activities \( e_i, i = 1, \ldots, n \), are measured in the same unit. The agent’s individual effort choice, however, is not observable, and thus cannot be specified in a court-enforceable contract.

As emphasized in the Introduction, we focus on a multitasking environment where the agent exhibits different skills to conduct the various tasks, which constitutes the key characteristic of our framework. To do so, we let \( \psi_i > 0 \) denote the agent’s ability with respect to performing task \( i \), where the diagonal \( n \times n \) matrix \( \Psi = \text{diag} (\psi_1, \ldots, \psi_n) \) summarizes these task-specific abilities. The agent’s disutility of implementing effort \( e \), denoted \( C(e) \), depends on his task-specific abilities \( \Psi \), and takes the form \( C(e) = e^T \Psi e / 2 \). Clearly, a higher ability for performing task \( i \) is reflected by a lower cost parameter \( \psi_i, i = 1, \ldots, n \), and vice versa.\(^10\)

The agent’s preferences are represented by the negative exponential utility function

\[
U(w, e) = -\exp \left[ -\rho (w - C(e)) \right],
\]

where \( \rho \) denotes the Arrow-Pratt measure of his absolute risk-aversion, and \( w \) as the agent’s wage. For parsimony, we let \( \bar{w} = 0 \) be the agent’s reservation wage, which provides him with the reservation utility \( \bar{U} = -1 \).

By implementing effort \( e \), the agent contributes to the principal’s gross payoff \( V(e) \), with

\[
V(e) = \mu^T e + \varepsilon_V,
\]

where \( \varepsilon_V \) is a normally distributed random variable with zero mean and variance \( \sigma^2_V \), representing firm-specific and economy wide risk. Moreover, the \( n \)-dimensional vector \( \mu = (\mu_1, \ldots, \mu_n)^T, \mu \in \mathbb{R}^n^+ \), characterizes the marginal effect of the agent’s effort \( e \) on the gross

\(^8\) For considerations on how multiple tasks are efficiently split among several agents, refer e.g. to Holmström and Milgrom (1991), Corts (2007), and Schöttner (2008).
\(^9\) All vectors are column vectors where ‘\( T \)’ denotes the transpose.
\(^10\) A similar approach is used by MacLeod (1996), where \( \psi_i, i = 1, \ldots, n \), are random variables. However, his work is different in the sense that he focuses on the relationship between explicit and implicit incentive contracts rather than on the effort distortion induced by imperfect performance measurements.
payoff $V(e)$. We assume that the agent’s contribution to firm value—reflected by $V(e)$—is too complex to be verifiable by outside parties and thus, cannot be part of a court-enforceable contract. The only contractible information about the agent’s specific effort choice $e$, however, is provided by the performance measure $P(e)$, with

$$P(e) = \omega^T e + \varepsilon,$$  

(3)

where the vector $\omega = (\omega_1, \ldots, \omega_n)^T$, $\omega \in \mathbb{R}^{n+}$, captures the marginal effect of the agent’s effort $e$ on the performance measure $P(e)$. Moreover, $\varepsilon$ is a normally distributed random variable with zero mean and variance $\sigma^2$, and represents potential effects on the performance measure beyond the agent’s control.

To motivate effort, the principal ties the agent’s compensation $w$ to his performance measure $P(e)$. In line with the previous multi-task agency literature, we restrict our analysis to a linear compensation scheme for the agent.$^{11}$ Specifically, the agent’s compensation $w$ takes the form

$$w(e) = \alpha + \beta P(e),$$  

(4)

where $\alpha$ denotes the base wage, and $\beta$ the incentive parameter. As well known, the base wage $\alpha$ can be used to split the surplus from the employment relationship between the principal and the agent, whereas adjusting the incentive parameter $\beta$ allows the principal to influence the agent’s specific effort choice. Since the compensation scheme $w$ is linear, the agent’s utility is exponential, and the error term of the performance measure $P(e)$ is normally distributed, maximizing the agent’s expected utility is equivalent to maximizing his certainty equivalent

$$CE(e) = \alpha + \beta \omega^T e - \frac{1}{2} e^T \Psi e - \frac{\rho}{2} \beta^2 \sigma^2,$$  

(5)

$^{11}$As shown by Holmström (1979) and Grossman and Hart (1983), optimal contracts are not necessarily linear. However, linear contracts are widely used in the economic literature, in particular in the multi-task agency literature, see e.g. Feltham and Xie (1994), Bushman et al. (2000), Banker and Thevaranjan (2000), Baker (2002), and Hughes, Zhang, and Xie (2005). The benefit of using linear compensation schemes in multi-task agency models is that they provide additional and clear insights into the agent’s choice for allocating his effort across multiple tasks. Nonetheless, for a continuous time model with Brownian motion where the agent controls the drift rate, Holmström and Milgrom (1987) found the optimal incentive scheme to be linear. Thus, we can interpret a single-period framework (as in this paper) as a one-shot consideration of the continuous time model à la Holmström and Milgrom (1987).
where $\rho \beta^2 \sigma^2 / 2$ is the risk premium required to compensate the risk-averse agent for the uncertainty in his expected income.

The timing of this problem is as follows. At date 0, the principal offers the agent an employment contract $(\alpha^*, \beta^*)$. If this contract guarantees at least the same expected utility as his reservation wage $\bar{w} = 0$, the agent accepts the contract and enters into the employment relationship with the principal. At date 1, after accepting his contract, the agent implements effort $e_i$ to perform task $i$, with $i = 1, \ldots, n$. At date 2, the agent’s performance measure $P(e)$ as well as his contribution to firm value, $V(e)$, are realized. At date 3, all payments are made.

Finally, to illustrate the fundamental forces in our multitasking framework, we need to briefly elaborate on the difference between the agent’s effort intensity and his effort allocation. Formally, let two arbitrary activities $e_k$ and $e_j$ change to $\hat{e}_k$ and $\hat{e}_j$, respectively. If the ratio between both activities remains identical such that $e_k/e_j = \hat{e}_k/\hat{e}_j$, $k, j = 1, \ldots, n, k \neq j$, the relative effort allocation does not change. In contrast, if $e_k/e_j \neq \hat{e}_k/\hat{e}_j$ for at least one pair $(k, j) \in \{1, \ldots, n\}, k \neq j$, the agent’s relative effort allocation varies. Intuitively, the agent now focuses more on (at least) one task relative to the other tasks. The overall effort intensity, however, changes without affecting the effort allocation, if there exists a constant $\lambda > 0$ satisfying $e = \lambda \hat{e}$, where $\hat{e}$ is the agent’s new effort vector.

### 3 The First-Best Effort Allocation

As a benchmark for our subsequent analysis, we first characterize the agent’s effort intensity and allocation which would maximize the principal’s expected profit. To do so, suppose the principal can specify a desired effort allocation and intensity in a court-enforceable employment contract. The optimal (i.e., first-best) effort vector, denoted $e^{fb}$, maximizes the difference between the agent’s expected contribution to firm value $E[V(e)]$ and his disutility $C(e)$ of implementing effort $e$:

$$
\max_e \Pi(e) = \mu^T e - \frac{1}{2} e^T \Psi e.
$$

(6)

For parsimony, let $\phi \equiv \Psi^{-1} \mu = (\mu_1/\psi_1, \ldots, \mu_n/\psi_n)^T$ denote the vector of the payoff-cost sensitivity ratios. Then, the first-best effort vector $e^{fb}$ is characterized by

$$
e^{fb} = \phi.
$$

(7)
The principal would maximize her expected profit by inducing the agent to perform each activity $e_i$, $i = 1, ..., n$, in accordance to its payoff-cost sensitivity ratio $\mu_i/\psi_i$. Clearly, activities with high ratios are more intensively performed in a first-best environment relative to activities with low ratios. This observation is quite intuitive: Whenever a task has a relatively high marginal contribution to firm value (i.e., $\mu_i$ is relatively high), or the agent has especially good skills to perform this task (i.e., $\psi_i$ is comparably low), the principal prefers the agent to focus relatively more on this particular task.

For the subsequent analysis keep in mind that the first-best effort vector $e^{fb}$ describes the efficient allocation of the agent’s effort across all relevant tasks. We henceforth refer to the first-best effort allocation as non-distorted effort. Any implemented effort vector $e^*$ characterized by a different effort allocation (compared to the first-best effort vector $e^{fb}$), is henceforth called distorted. Formally, effort vector $e^*$ is distorted if there exists no constant $\lambda > 0$ satisfying $e^* = \lambda e^{fb}$.

### 4 The Second-Best Contract

If the principal cannot specify a desired effort vector $e$ in a court-enforceable employment contract, she faces an incentive problem. Specifically, the principal needs to tie the agent’s compensation to the only contractible information about his effort choice $e$: the (imprecise) performance measure $P(e)$. The application of the performance measure $P(e)$ in the agent’s incentive contract, however, can cause the following two inefficiencies. First, the performance measure $P(e)$ comprises some noise (captured by the random variable $\varepsilon$) such that the risk-averse agent requires a risk premium for accepting a contract dependent on $P(e)$. Second, the performance measure $P(e)$ does not necessarily capture the relative importance of the agent’s tasks with respect to the firm value $V(e)$. In such a situation, utilizing the performance measure to provide the agent with effort incentives motives him to inefficiently allocate his effort across the relevant tasks (see e.g. Feltham and Xie (1994) and Baker (2002)). Technically, this is the case whenever the vector $\mu$ from the firm’s payoff function $V(e)$ and the vector $\omega$ from the agent’s performance measure $P(e)$ are linearly independent, i.e., whenever there does not exist a constant $\lambda \neq 0$ which satisfies $\mu = \lambda \omega$. Such a performance measure is henceforth referred to as incongruent.
Baker (2002) derived a very intuitive geometric measure for performance measure congruity. Since we will make extensive use of this simple measure throughout this paper, we summarize Baker’s result in the following definition.

**Definition 1.** The congruence of performance measure \( P(e) \) is measured by \( \Upsilon_C(\varphi) = \cos \varphi \), where \( \varphi \) is the angle between the vector of payoff sensitivities \( \mu \) and the vector of performance measure sensitivities \( \omega \).

According to Definition 1, performance measure \( P(e) \) is incongruent if vector \( \mu \) and vector \( \omega \) are linearly independent, which in turn implies that \( \varphi \neq 0 \). Moreover, a more congruent performance measure is characterized by a smaller angle \( \varphi \), and hence, implies a higher measure of congruity \( \Upsilon_C(\varphi) \) due to the definition of the cosine. Finally note that \( \varphi \in [0, \pi/2] \) as \( \mu_i, \omega_i \geq 0, i = 1, ..., n \).

In a second-best environment, the principal’s problem is to design a contract \((\alpha^*, \beta^*)\) that maximizes her expected profit \( \Pi = E[V(e) - w(e)] \), while ensuring the agent’s participation in this employment relationship. The optimal linear contract therefore solves

\[
\max_{\alpha, \beta, e} \Pi \equiv \mu^T e - \alpha - \beta \omega^T e \tag{8}
\]

s.t.

\[
e = \arg \max_{\tilde{e}} \alpha + \beta \omega^T \tilde{e} - \frac{1}{2} \tilde{e}^T \Psi \tilde{e} - \frac{\rho}{2} \beta^2 \sigma^2 \tag{9}
\]

\[
\alpha + \beta \omega^T e - \frac{1}{2} e^T \Psi e - \frac{\rho}{2} \beta^2 \sigma^2 \geq 0, \tag{10}
\]

where (9) is the agent’s incentive constraint and (10) his participation constraint.

To simplify our subsequent analysis, we define \( \Gamma \equiv \Psi^{-1} \omega = (\omega_1/\psi_1, ..., \omega_n/\psi_n)^T \) as the vector of measure-cost sensitivity ratios. We can then infer from (9) that the agent—for any given contract \((\alpha, \beta)\)—implements the effort vector \( e^* \) with

\[
e^* = \Psi^{-1} \omega \beta = \Gamma \beta. \tag{11}
\]

In contrast to the first-best effort \( e^{fb} \) as characterized in Section 3, the agent’s effort \( e^*_i \) for performing task \( i \) now depends on the measure-cost sensitivity ratio \( \omega_i/\psi_i \) as well as on the incentive parameter \( \beta \). It becomes clear from (11) that the principal can adjust the incentive

\[\text{All angles are represented in radian measures throughout this paper.}\]
parameter $\beta$ to influence the agent’s effort intensity. However, the allocation of the agent’s effort across the relevant tasks is exogenously determined by the performance measure sensitivities $\omega$ relative to the agent’s task-specific abilities as captured by $\Psi$, and is therefore beyond the control of the principal.

To maximize her expected profit, the principal sets the base wage $\alpha$ such that the agent’s participation constraint (10) becomes binding. Solving (10) for $\alpha$ and substituting the resulting expression together with $e^*$ in the principal’s objective function (8) yield the following unconstrained maximization problem for the principal:

$$\max \beta \quad \Pi = \mu^T \Gamma \beta - \frac{\beta^2}{2} [\omega^T \Gamma + \rho \sigma^2].$$

(12)

The first-order condition leads to the optimal incentive parameter $\beta^*$:

$$\beta^* = \frac{\mu^T \Gamma}{\omega^T \Gamma + \rho \sigma^2}.\quad (13)$$

Besides the precision of the performance measure $1/\sigma^2$ with the agent’s risk tolerance $1/\rho$, the optimal incentive parameter $\beta^*$ is clearly a function of the payoff sensitivities $\mu$, the performance measure sensitivities $\omega$, and the measure-cost sensitivity ratios $\Gamma$. Recall that $\Gamma = \Psi^{-1} \omega$. This observation, in turn, implies that the optimal incentive parameter $\beta^*$ accounts for the agent’s task-specific abilities—as reflected by $\Psi$—in two different ways: (i) by their relations to the payoff sensitivities $\mu$ in the numerator; and (ii), by their relations to the performance measure sensitivities $\omega$ in the numerator and denominator of $\beta^*$. We can thus infer that agents with different task-specific abilities receive diverse incentive contracts even if in charge of performing an identical set of tasks, and if evaluated based on the same information system. Put differently, incentive contracts in multi-task environments are not simply adjusted to the characteristics of available performance measures, but are also to the specific abilities of agents to perform the relevant tasks.

We can now derive the principal’s expected profit under the optimal incentive contract $(\alpha^*, \beta^*)$. Substituting $\beta^*$ in the principal’s objective function (12) yields

$$\Pi^* = \frac{(\mu^T \Gamma)^2}{2 (\omega^T \Gamma + \rho \sigma^2)}.\quad (14)$$

Using geometric representations for the vector products $\mu^T \Gamma$ and $\omega^T \Gamma$, we can rewrite the principal’s expected profit $\Pi^*$ as follows:

$$\Pi^* = \frac{\|\mu\|^2 \|\Gamma\|^2 \cos^2 \theta}{2 (\|\omega\| \|\Gamma\| \cos \xi + \rho \sigma^2)},$$

(15)
where $\theta$ denotes the angle between the vector of payoff sensitivities $\mu$ and the vector of measure-cost sensitivity ratios $\Gamma$.\textsuperscript{13} The angle between the vector of performance measure sensitivities $\omega$ and vector $\Gamma$ is denoted by $\xi$. As will be shown in the next section, both angles $\theta$ and $\xi$ can be used to characterize the efficiency of the agent’s induced effort allocation, and as a logical consequence, its effect on the principal’s expected profit $\Pi^\ast$.

5 Performance Measure Congruity and Effort Distortion

In this section, we focus more intensively on performance measure congruity and its effect on effort distortion when the agent performs different tasks with varying degrees of ease. To do so, we first need to clarify the distinction between performance measure congruity and effort distortion. Performance measure congruity refers to the degree of alignment between the agent’s marginal effect on his performance measure and his marginal effect on the expected payoff for the firm (Feltham and Xie, 1994). Performance measure congruity can thus be characterized by the angle $\varphi$ between the vector of payoff sensitivities $\mu$ and the vector of performance measure sensitivities $\omega$, as emphasized by Baker (2002) and summarized by Definition 1 in Section 2.

On the other hand, effort distortion refers to the relationship between the agent’s specific multidimensional effort choice $e$ and the first-best effort vector $e^{fb}$. Formally, as emphasized in Section 3, the agent’s specific effort choice $e^\ast$ in a second-best environment is non-distorted if there exists a constant $\lambda > 0$ satisfying $e^{fb} = \lambda e^\ast$. Recall that $e^{fb} = \Psi^{-1} \mu$ and $e^\ast = \beta \Psi^{-1} \omega$. Thus, we can immediately infer that only a perfectly congruent performance measure motivates the agent to implement non-distorted effort. Technically, the principal can only induce non-distorted effort if $\mu = \lambda \omega$ holds for a constant $\lambda \in \mathbb{R}^+$, which implies that $\varphi = 0$ (see Definition 1). Notice that this conclusion does not depend on the agent’s task-specific abilities as captured by $\Psi$. Consequently, the observation of Feltham and Xie (1994)—that only congruent performance measures induce non-distorted effort—holds even in a more general setting where the agent is allowed to have different abilities to perform the relevant tasks. On the other hand, if the performance measure is incongruent, we can conclude that its application in the agent’s incentive contract motivates him to implement a distorted effort allocation, which is clearly inefficient from the perspective of the principal.

\textsuperscript{13}We follow the convention and use the notation $\| \cdot \|$ to refer to the length of a vector.
The objective of the subsequent analysis is to characterize the degree of effort distortion induced by the application of incongruent performance measures, and to investigate how effort distortion is affected by the agent’s task-specific abilities. To unravel the effects of the agent’s task-specific abilities on the efficiency of his incentive contract, we henceforth assume that for at least two activities $e_k$ and $e_j$ the respective abilities of the agent are not identical. Formally, $\psi_k \neq \psi_j$ for at least one pair $(k, j) \in \{1, \ldots, n\}$, $k \neq j$, which implies that vector $\mathbf{\Gamma}$ is linearly independent of vector $\mathbf{\omega}$. To illustrate the fundamental forces, we first discuss the economic interpretations of the two angles $\theta$ and $\xi$, which apparently determine the principal’s expected profit $\Pi^*$, and thus, can expected to be rooted in the agent’s specific effort choice.

**Proposition 1.** If $\psi_k \neq \psi_j$ for at least one pair $(k, j) \in \{1, \ldots, n\}$, $k \neq j$, then $\Upsilon_D(\theta) = \cos \theta$ measures the induced effort distortion.

**Proof** All proofs are given in the Appendix.

Proposition 1 exposes an important result: Whenever the agent exhibits different skills to perform the relevant tasks, the distortion of his induced effort allocation—relative to the first-best effort allocation—is measured by the angle $\theta$ between the vector of payoff sensitivities $\mu$ and the vector of measure-cost sensitivity ratios $\mathbf{\Gamma}$. Note, however, that the measure $\Upsilon_D(\theta)$ is negatively related to effort distortion due to the definition of the cosine. More precisely, the less distorted the agent’s effort allocation, the smaller is $\theta$, and consequently, the higher is $\Upsilon_D(\theta)$. If $\theta = 0$, the application of performance measure $P(e)$ in the agent’s incentive contract motivates him to implement a non-distorted (i.e., first-best) effort allocation. According to our previous discussion, however, this requires a perfectly congruent performance measure (i.e., $\mu = \lambda \mathbf{\omega}$ with $\lambda \in \mathbb{R}^+$).

For illustrative purposes, suppose for a moment that the only available performance measure $P(e)$ changes such that it induces the agent to implement a less distorted effort allocation. Formally, $\theta$ decreases. We can infer from (15) (see Section 4) that this, *ceteris paribus*, leads to a higher expected profit $\Pi^*$ for the principal. Note, however, that there is a second effect on $\Pi^*$ captured by $\xi$ as the angle between the vector of performance measure sensitivities $\omega$ and the vector of measure-cost sensitivity ratios $\mathbf{\Gamma}$. To illustrate this effect, we can rewrite the agent’s cost of effort $C(e)$ by substituting $e^*$:

$$C(\cdot) = \frac{1}{2} \beta^2 \|\omega\| \|\mathbf{\Gamma}\| \cos \xi.$$ (16)
Observe that the agent’s task-specific abilities affect his cost of effort $C(e)$ in two different ways. The first effect is a result of his effort cost intensity over all tasks. To see this, assume for a moment that the agent’s cost of effort takes the form $C(e) = e^T \lambda \Psi e / 2$ with $\lambda > 0$. Increasing $\lambda$ implies that it becomes more costly for the agent to perform all relevant tasks, thereby increasing the length of vector $\Gamma$ (i.e., $\|\Gamma\|$) without affecting the angle $\cos \xi$.

The second effect is rooted in the relationship between the performance measure sensitivities $\omega$ and the agent’s task-specific abilities $\Psi$. More specifically, the definition of vector products suggests that changing the agent’s relative abilities across tasks does not only affect the length of vector $\Gamma$ (i.e., $\|\Gamma\|$), but also the angle $\xi$. Recall that $\|\Gamma\|$ characterizes only the agent’s overall effort intensity, and not his relative effort allocation. We can therefore conclude that the angle $\xi$ measures the agent’s cost of effort (in utility terms) for a particular effort allocation motivated by the use of performance measure $P(e)$, henceforth referred to as measure-cost efficiency. Since this observation is fundamental for our subsequent analysis, it is summarized by the next proposition.

**Proposition 2.** If $\psi_k \neq \psi_j$ for at least one pair $(k, j) \in \{1, ..., n\}$, $k \neq j$, then $\Upsilon^{M/C}(\xi) = \cos \xi$ characterizes the measure-cost efficiency.

Our previous results are illustrated in Figure 1 for the three-dimensional case ($n = 3$). Besides the agent’s second-best effort vector $e^*$, Figure 1 depicts the vectors of the payoff sensitivities $\mu$, performance measure sensitivities $\omega$, and measure-cost sensitivity ratios $\Gamma$. Observe that the agent’s effort vector $e^*$ has the same direction as the vector of measure-cost sensitivity...
ratios $\Gamma$; only their lengths differ, which clearly depends on the specific incentive parameter $\beta$. Moreover, notice that the effort vector $e^*$ is not necessarily on the plane spanned by vectors $\mu$ and $\omega$. By drawing on our previous observations, we can infer that the location of effort vector $e^*$ relative to the vector of the payoff sensibilities $\mu$ characterizes the induced effort distortion (measured by the angle $\theta$). On the other hand, the relationship between the vector of payoff sensitivities $\mu$ and the vector of performance measure sensitivities $\omega$ measures the congruity of performance measure $P(e)$, see Definition 1 (captured by the angle $\varphi$). Finally, the measure-cost efficiency is characterized by the relation between the vectors $\Gamma$ and $\omega$ (measured by angle $\xi$).

To gain additional insights, we now discuss a special case which is not illustrated by Figure 1: Vector $\mu$ and vector $\omega$ point in the same direction. In this case, as emphasized above, performance measure $P(e)$ is perfectly congruent, which means that $\varphi = 0$; see Definition 1. This, on the other hand, implies that the agent’s incentive contract—based on the congruent performance measure $P(e)$—motivates him to implement the non-distorted effort allocation. Formally, $e^{fb} = \lambda e^*$ with $\lambda > 0$ when $\varphi = 0$. While the principal can then induce the first-best effort allocation, motivating the first-best effort intensity by adjusting the incentive parameter $\beta$ is not necessarily optimal. As well known, inducing the first-best effort intensity can only be optimal from the principal’s perspective, if either the agent is risk-neutral, or the available performance measure is perfectly precise (i.e., $\sigma^2 = 0$). Otherwise, the principal would impose too much risk an the risk-averse agent, which would require the payment of a higher risk-premium to ensure his participation in the employment relationship.

We now briefly consider the case where the agent has identical abilities for performing all relevant tasks, i.e., $\psi_i = \hat{\psi} > 0$, $i = 1, ..., n$. The vector of measure-cost sensitivity ratios $\Gamma$ then becomes $\Gamma = \omega/\hat{\psi}$, so that vector $\Gamma$ and vector $\omega$ now point in the same direction. This further implies that the agent’s second-best effort vector $e^*$ becomes $e^* = \omega \beta/\hat{\psi}$. Thus, the effort vector $e^*$ and the measure sensitivity vector $\omega$ point in the same direction; only their lengths differ, which depends on the specific incentive parameter $\beta$ and the effort cost parameter $\hat{\psi}$ (which is identical for all tasks). We can then infer that the measure of performance measure congruity ($\Upsilon^C(\varphi)$) is now identical to the measure of effort distortion ($\Upsilon^D(\theta)$). This important observation is summarized by the next proposition, and formally proved in the Appendix.

**Proposition 3.** If $\psi_i = \hat{\psi} > 0$, $i = 1, ..., n$, then $\Upsilon^D(\varphi) = \Upsilon^C(\varphi) = \cos \varphi$. 


Proposition 3 highlights a fundamental result: If agents do not exhibit different task-specific abilities, as commonly assumed in the previous multi-task agency literature, performance measure congruity and effort distortion are captured by the same measure. However, if agents do possess different abilities for performing the relevant tasks, it becomes crucial to distinguish between the concepts of performance measure congruity and effort distortion. As shown, the application of incongruent performance measures in incentive contracts results in effort distortion, and the extent of this inefficiency is determined by the agent’s task-specific abilities.

To summarize our previous observations, consider again the principal’s expected profit $\Pi^*$ as stated in Section 4. According to our previous discussion, $\Pi^*$ consists of three components: (i) the measure of effort distortion $\Upsilon^D(\theta)$ in the numerator, (ii) the measure-cost efficiency $\Upsilon^{M/C}(\xi)$ in the denominator; and (iii), the measure of the agent’s risk aversion $\rho$ in conjunction with the variance $\sigma^2$ of the applied performance measure $P(e)$ in the denominator. It is well known that the trade-off between incentive risk and the agent’s desire for insurance affects the design, and therefore the efficiency, of incentive contracts. Moreover, as demonstrated by Feltham and Xie (1994) and Baker (2002), optimal incentive contracts in multi-task agency relationships additionally account for the degree of congruity of available performance measures. However, when agents possess task-specific abilities, our analysis reveals that the measure-cost efficiency constitutes a third crucial factor for the optimal design of incentive contracts in multi-task agencies.

6 Ranking Performance Measures

As Feltham and Xie (1994) emphasized, performance measures may differ with respect to their congruity as well as their precision. Our previous analysis revealed that task-specific abilities additionally play a crucial role for the contract efficiency. To gain additional insights, we focus in this section on how the attributes of performance measures as well as characteristics of agents determine the relative value of contractible measures in multi-task agency relationships.

Consider a set $P \subseteq \mathbb{R}^m$ of $m \geq 2$ performance measures with $P_i(e) \in P$, and

$$P_i(e) = \omega_i^T e + \varepsilon_i, \quad (17)$$

where the random variable $\varepsilon_i$ is normally distributed with zero mean and standard deviation
To illustrate the relative value of every individual performance measure, we can simply compare the expected profits for the principal each of these measures would generate if applied in the agent’s incentive contract. Then, performance measure $P_k(e)$ is referred to be strictly superior if it generates a higher expected profit for the principal than all other available performance measures $P_i(e) \in P, i \neq k$.

For single-task agency relationships, Kim and Suh (1991) have shown that the value of individual performance measures can be compared by their respective signal-to-noise ratio. In single-task agency relationships, performance measures with higher signal-to-noise ratios provide more precise information about the agent’s specific effort choice, and are therefore strictly preferred to performance measures with lower signal-to-noise ratios. Schnedler (2008) generalized the signal-to-noise ratio to account for multidimensional effort. According to Schnedler (2008) (see Definition 2), the signal-to-noise ratio of performance measure $P_i(e)$, denoted $\Lambda_i$, is

$$\Lambda_i = \frac{\nabla P_i(e)^T \nabla P_i(e)}{\sigma_i^2},$$  

where $\nabla P_i(e)$ is the gradient of performance measure $P_i(e)$ with respect to the effort vector $e$. Applied to our setting, the signal-to-noise ratio $\Lambda_i$ for performance measure $P_i(e)$ is

$$\Lambda_i = \frac{\|\omega_i\|^2}{\sigma_i^2}.  \eqno (19)$$

Clearly, the signal-to-noise ratio does not account for the agent’s task-specific abilities even though—as shown in Section 5—the agent’s abilities affect the principal’s expected profit $\Pi^*$. We can therefore immediately infer that signal-to-noise ratios are generally not sufficient to rank performance measures in multi-task agency relationships, especially when agents possess different task-specific abilities. This conclusion is supported by the next proposition.

**Proposition 4.** Performance measure $P_k(e)$ is strictly superior to any other performance measure $P_j(e) \in P, i = 1, ..., m, j \neq k$, if and only if,

$$\frac{\|\omega_k\| \rho \sigma_k^2}{\|\Gamma_k\| (\hat{\Gamma}_k^D(\theta_k))^2} < \frac{\|\omega_j\| \rho \sigma_j^2}{\|\Gamma_j\| (\hat{\Gamma}_j^D(\theta_j))^2},  \eqno (20)$$

Subscript $i$ henceforth refers to performance measure $P_i(e) \in P$.

Suppose that the performance measure $P_i(e)$ takes the form $P_i(e) = \omega_i e + \epsilon_i$, where effort $e$ and the performance measure sensitivity $\omega_i$ are now one-dimensional. Since the random variable $\epsilon_i$ is normally distributed with variance $\sigma_i^2$, the signal-to-noise ratio of performance measure $P_i(e)$ is then given by $\omega_i^2/\sigma_i^2$; see Kim and Suh (1991).
where $\Upsilon^D_i(\theta_i)$ is the measure of distortion induced by performance measure $P_i(e)$, and $\Upsilon^{M/C}_i(\xi_i)$ the corresponding measure-cost efficiency, with $i = \{j, k\}$.

**Proof** Follows directly from rearranging $\Pi^*(P_k(e)) > \Pi^*(P_j(e))$ and substituting $\Upsilon^{M/C}_i(\xi_i) = \cos \xi_i$ and $\Upsilon^D_i(\theta_i) = \cos \theta_i$, $i = k, j$.

Proposition 4 provides important insights. Specifically, the relative value of a performance measure in a multitasking environment depends on two ratios: (i) the normalized ratio between the measure-cost efficiency $\Upsilon^{M/C}_i(\xi_i)$ and the induced effort distortion $\Upsilon^D_i(\theta_i)$; and (ii), the normalized inverse of the effort distortion measure $\Upsilon^D_i(\theta_i)$ in relation to the variance $\sigma^2_i$ of the performance measure and the agent’s degree of risk-aversion $\rho$. Finally, observe that performance measure congruity—as measured by the angle $\phi$—does not directly enter the ranking criterion as emphasized by Proposition 4. It does, however, indirectly affect the measure of effort distortion $\Upsilon^D_i(\theta_i)$ as well as the measure-cost efficiency as characterized by $\Upsilon^{M/C}_i(\xi_i)$.

Our previous analysis suggests that the value of performance measures in multi-task agency relationships cannot necessarily be measured by their respective signal-to-noise ratios. As shown, it is indispensable to take the induced effort distortion as well as the measure-cost efficiency into consideration—both determined by the agent’s task-specific abilities as reflected by the diagonal matrix $\Psi$. This observation has a very important implication: Comparing the values of individual performance measures in multi-task agency relations necessitates specific knowledge about the agent’s characteristics, which is not required for ranking performance measures in single-task principal-agent relationships. Therefore, observations from single-task agency models cannot necessarily be transferred to a more general setting, where the agent is not only in charge of multiple tasks, but also exhibits different skills to perform these tasks.

In the remainder of this section, we identify conditions which allow us to define a ranking criterion for performance measures in multi-task agency relationships independent of the specific characteristics of agents. To do so, we first abstract away from task-specific abilities of agents. Formally, we now assume that $\psi_i = \hat{\psi} > 0$, $i = 1, ..., n$. The next corollary to Proposition 4 exposes an adjusted ranking criterion which is applicable in multi-task agency relations in the absence of task-specific abilities of agents.

**Corollary 1.** Suppose that $\psi_i = \hat{\psi} > 0$, $i = 1, ..., n$. Then, performance measure $P_k(e)$ is strictly superior to any other performance measure $P_j(e) \in P$, $i = 1, ..., m$, $j \neq k$, if and only
if,
\[
\frac{1}{\gamma^C_k(\varphi_k)} \left[ 1 + \hat{\psi}_k \Lambda_k^{-1} \right]^{\frac{1}{2}} < \frac{1}{\gamma^C_j(\varphi_j)} \left[ 1 + \hat{\psi}_j \Lambda_j^{-1} \right]^{\frac{1}{2}},
\]
where \(\Lambda_i\) is the signal-to-noise ratio of performance measure \(P_i(e)\), and \(\gamma^C_i(\varphi_i)\) its measure of congruity, with \(i = \{j, k\}\).

According to Corollary 1, we can use adjusted signal-to-noise ratios to rank performance measures in multi-task agency relationships, but only if the agent has identical skills for all relevant tasks. In this case, the agent’s specific effort allocation depends exclusively on the characteristics of his performance evaluation. Nonetheless, it is still indispensable to know the agent’s effort cost parameter \(\hat{\psi}\) as well as his degree of risk-aversion \(\rho\) in order to assess the relative value of performance measures.

The next proposition provides sufficient conditions which ensure that performance measures in multi-task agency relations can be ranked exclusively by their respective signal-to-noise ratios.

**Proposition 5.** Suppose that \(\psi_i = \hat{\psi} > 0, i = 1, \ldots, n\), and that there exist constants \(\lambda_j \neq 0\) which satisfy \(\omega_i = \lambda_j \omega_j\) for all \(i, j = 1, \ldots, m, i \neq j\). Then, performance measure \(P_k(e)\) is strictly superior to any other performance measure \(P_j(e) \in P, i = 1, \ldots, m, j \neq k\), if and only if \(\Lambda_k > \Lambda_j\).

Proposition 5 implies that signal-to-noise ratios are sufficient to rank performance measures in multi-task agency relations if the following two conditions are satisfied: First, the agent has identical skills for all relevant tasks (i.e., \(\psi_i = \hat{\psi} > 0, i = 1, \ldots, n\)). Second, all performance measures provide the same information about the agent’s relative effort allocation (i.e., \(\omega_i = \lambda_j \omega_j\) with \(\lambda_j \neq 0\) for all \(i, j = 1, \ldots, m\)). Notice that the latter condition also implies that \(\gamma^C_i(\varphi_i) = \gamma^C_j(\varphi_j), i, j = 1, \ldots, m\), i.e., all performance measures share the same measure of congruity.\(^{16}\) If both conditions are satisfied, each performance measure—if applied in the agent’s incentive contract—would result in the same effort distortion, and would lead to the same measure-cost efficiency. The relative value of performance measures is then solely defined

\(^{16}\)Note, however, that the reverse inference cannot be made. That is, if \(\gamma^C_i(\varphi_i) = \gamma^C_j(\varphi_j)\), it is not necessarily true that \(\omega_i = \lambda_j \omega_j\) holds with \(\lambda_j \neq 0, i, j = 1, \ldots, m, i \neq j\). In this case, the signal-to-noise ratio is not sufficient to rank performance measures in multi-task principal-agent relationships.
by their respective precision (i.e., $1/\sigma_i^2$) and scale (i.e., $\|\omega_i\|$)—both captured by their respective signal-to-noise ratio $\Lambda_i$.

To shed more light on the influence of task-specific abilities of agents on the relative value of performance measures in multi-task agency relations, we can eliminate potential effects related to their precisions $1/\sigma_i^2$. By letting $\sigma_i^2 = 0$, condition (20) from Proposition 4 simplifies to

$$
\nu \frac{\cos^2 \theta_k}{\cos^2 \theta_j} > \frac{\cos \xi_k}{\cos \xi_j}, \quad \nu = \frac{\|\omega_j\| \|\Gamma_k\|}{\|\omega_k\| \|\Gamma_j\|}.
$$

(22)

It now becomes clearer from (22) that the relative value of performance measures depends—besides on their respective precision and scale—on the induced effort distortions ($\cos \theta_i$) and the respective measure-cost efficiency ($\cos \xi_i$), $i = k, j$, weighted by the multiplier $\nu$.$^{17}$ Put differently, in the special case where either the agent is risk-neutral (i.e., $\rho = 0$) or performance measures are not influenced by factors beyond the agent’s control (i.e., $\epsilon_i = 0$), the relative value of performance measures hinges on the following two factors: (i) the respective efficiency of the motivated effort allocation ($\cos \theta_i$); and (ii) the agent’s disutility of effort which stems from his specific effort allocation ($\cos \xi_i$).

7 Discussion

The analysis in this paper shows that agents’ task-specific abilities play a crucial role for the optimal design of incentive contracts, and thus, for the value of employment relationships from the perspective of the principal. This observation, in turn, clearly accentuates the importance of an adequate job design as well as job assignment for the efficiency of an organization. In this section, we provide a brief and intuitive discussion of the main implications for the efficient job design and job assignment when agents are characterized by different skills and abilities for performing the relevant tasks.

For illustrative purposes, consider two organizations (or divisions within these organizations) with different information systems (i.e., different performance measures). Our analysis suggests that these organizations (or divisions) may prefer agents with different sets of task-specific abilities even if their potential contributions to firm value are identical. This implication

$^{17}$Note that the multiplier $\nu$ normalizes the scales $\|\omega_i\|, i = k, j$, of both performance measures in order to make them comparable.
is clearly rooted in the revealed distortion effect and measure-cost efficiency effect. A similar observation can be made when agents’ potential contributions differ across organizations (or divisions), but these agents are evaluated by equivalent performance measures—which stems from the distortion effect.

Next, consider a manager and a worker who, for simplicity, share the same risk tolerance. Due to his prior learning experience, the manager is assumed to exhibit relatively more abilities in performing administrative tasks than in conducting manufacturing related tasks. The reverse relationship is assumed for the worker. Now, who is more desirable from the perspective of the firm? As emphasized in this paper, this cannot be assessed without considering the specific characteristics of the relevant job. Based on our assumptions with respect to their task-specific abilities, the manager is clearly more suitable for jobs which consist primarily of administrative tasks, whereas it is more efficient to employ the worker for conducting mainly manufacturing related tasks. As a result, both the manager and the worker should be assigned to different jobs, and should receive various inventive contracts which are adjusted to their respective abilities and to the specific characteristics of their jobs.

Finally, suppose a firm wants to employ two specific managers $A$ and $B$ who, again for simplicity, share the same risk tolerance. Assume that manager $A$ exhibits a relatively higher ability in performing administrative tasks compared to manager $B$, but the latter can supervise subordinates more effectively. In this case, we can infer from our analysis that the firm adjusts the respective incentive contracts according to the managers’ individual abilities to perform the relevant tasks. As a result, both managers receive different incentive contracts even though they are in charge of conducting identical tasks.

In conclusion, the analysis in this paper suggests that the presence of task-specific abilities of agents has a fundamental effect on the optimal design of incentive contracts, and on the optimal assignment of employees to various jobs within firms.

8 Conclusion

As well known, incongruent performance measures—measures which do not perfectly reflect the multidimensional contributions of agents to firm value—motivate agents to implement inefficient effort allocations when applied in incentive contracts (see e.g. Feltham and Xie (1994)).
In this paper, we have extended the standard multi-task principal-agent model with incongruent performance measures to account for different skills and abilities of an agent to perform the relevant tasks. As we have shown, incentive contracts are adjusted to these task-specific abilities, and depend—besides on the precision and congruity of performance measures and the degree of the agent’s risk-aversion—on the following two factors: (i) the degree of the induced effort distortion which is determined by the agent’s task-specific abilities (*distortion effect*); and (ii), the agent’s disutility of effort which stems from his specific effort allocation (*measure-cost efficiency*).

This paper further investigated the relative value of multiple performance measures in multi-task agency relationships. One important implication to be drawn from our analysis is that the signal-to-noise ratio—commonly used to assess the relative value of performance measures in single-task agency relations—is not a sufficient ranking criterion in multi-task principal-agent relationships. As shown, the relative value of performance measures hinges on the task-specific abilities of agents, and it therefore closely tied to individual agents. This, in turn, provides the principal with some latitude to improve the efficiency of the organization by assigning agents to jobs they are most suited for. In conclusion, the presence of task-specific abilities can explain why different agents are allocated to various jobs; and why some agents receive different incentive contracts even if their jobs are identical.

Previous multi-task agency literature focused primarily on performance measure congruity and its effect on incentive contracts. As shown in this paper, we can shed more light on the nature of incentive contracts in multi-task agency relationships when keeping in mind that agents may differ in their skills and abilities to perform particular tasks. We note, however, that our model takes a somewhat simplified view of employment relationships insofar that agents’ abilities are perfectly known to the principal. One fruitful area for future research therefore concerns the optimal design of incentive contracts when the abilities and skills of agents are their private information.
9 Appendix

Proof of Proposition 1.
Effort distortion refers to the relation of \( e^* \) to \( \mu \) and can be therefore measured by the vector product \( \mu^T e^* \). Since \( e^* = \Gamma \beta \),
\[
\mu^T e = \beta \sum_{i=1}^{n} \mu_i \Gamma_i = \beta \| \mu \| \| \Gamma \| \cos \theta.
\] (23)
First note that \( \| \mu \| \) does not affect the relative importance of tasks for \( V(e) \). Furthermore, \( \beta \| \Gamma \| \) determines the lengths of vector \( e^* \), but not its direction in the \( n \)-dimensional space. The length is arbitrary in the sense that it can be adjusted by \( \beta \). Consequently, \( \Upsilon^D(\theta) = \cos \theta \in [0, 1] \) measures the induced effort distortion. \( \square \)

Proof of Proposition 3.
To measure effort distortion, we can use the vector product \( \mu^T e^* \). If \( \psi_i = \hat{\psi} > 0, \ i = 1, \ldots, n \), then \( e^* = \beta \omega / \hat{\psi} \). This leads to
\[
\mu^T e = \frac{\beta}{\hat{\psi}} \sum_{i=1}^{n} \mu_i \omega_i = \frac{\beta}{\hat{\psi}} \| \mu \| \| \omega \| \cos \varphi.
\] (24)
Again, \( \| \mu \| \) does not affect the relative importance of tasks for \( V(e) \), and \( \beta \| \omega \| \) determines the length of vector \( e^* \), but not its direction in the \( n \)-dimensional space. Thus, \( \bar{\Upsilon}^D(\varphi) = \cos \varphi \in [0, 1] \) measures distortion if \( \psi_i = \hat{\psi} > 0, \ i = 1, \ldots, n \). Consequently, \( \bar{\Upsilon}^D(\varphi) = \Upsilon^C(\varphi) \). \( \square \)

Proof of Corollary 1.
If \( \psi_i = \hat{\psi} > 0, \ i = 1, \ldots, n \), then \( \Gamma_i = \omega_i / \hat{\psi} \) and \( \| \Gamma_i \| = \| \omega_i \| / \hat{\psi}, \ i = \{j,k\} \). Consequently, \( \Upsilon_i^{M/C}(\xi_i = 0) = 1 \) and \( \Upsilon_i^D(\varphi_i) = \Upsilon_i^C(\varphi_i) \); see Proposition 3. By substituting \( \Lambda_i = \| \omega_i \|^2 / \sigma_i^2, \ i = \{j,k\} \), the ranking criterion of Proposition 4 can be rewritten as stated in the corollary. \( \square \)

Proof of Proposition 5.
Observe first that the principal’s expected profit \( \Pi^*(P(e^*)) \) on the basis of \( P_i(e^*) \) can be written as
\[
\Pi^*(P(e^*)) = \frac{(\mu^T \Gamma_i)^2}{2(\omega_i^T \Gamma_i + \rho \sigma_i^2)}. \tag{25}
\]
Recall that $\Gamma_i = \Psi^{-1}\omega_i$. Consequently, performance measure $P_k(e)$ is strictly superior to any other performance measure $P_j(e) \in \mathbf{P}$, $\forall j \neq k$, if and only if,

$$\frac{(\mu^T\Psi^{-1}\omega_k)^2}{2(\omega_k^T\Psi^{-1}\omega_k + \rho \sigma_k^2)} > \frac{(\mu^T\Psi^{-1}\omega_j)^2}{2(\omega_j^T\Psi^{-1}\omega_j + \rho \sigma_j^2)}.$$

(26)

If $\omega_k = \lambda \omega_j$, we can re-scale $P_j(e)$ such that it has the same sensitivity in $e$ as $P_k(e)$. Accordingly,

$$\bar{P}_j(e) = \omega_j^T e + \frac{\varepsilon_j}{\lambda},$$

(27)

where $\text{Var} [\bar{P}_j(e)] = \sigma_j^2 \lambda^{-2}$. Let $\omega \equiv \omega_i$, $i = j, k$. This leads to

$$\frac{(\mu^T\Psi^{-1}\omega)^2}{2(\omega^T\Psi^{-1}\omega + \rho \sigma_k^2)} > \frac{(\mu^T\Psi^{-1}\omega)^2}{2(\omega^T\Psi^{-1}\omega + \rho \sigma_j^2 \lambda^{-2})},$$

(28)

which can be re-arranged to

$$\frac{1}{\sigma_k^2} > \frac{\lambda^2}{\sigma_j^2}.$$

(29)

Recall that after re-scaling, $\omega_k = \omega_j$. Thus, (29) can be written as

$$\frac{\|\omega_k\|^2}{\sigma_k^2} > \frac{\lambda^2\|\omega_j\|^2}{\sigma_j^2},$$

(30)

which is equivalent to $\Lambda_k > \Lambda_j$. \qed
References


